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Ph.D. in Mathematical Physics

Thesis

Normal Matrix Models and Orthogonal Polynomials for a class of potentials with discrete rotational symmetries

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Abstract

In this thesis we are going to study normal random matrix models which generalize naturally the polynomially perturbed Ginibre ensamble considered for example in [18, 12, 11, 6], focusing in particular on their eigenvalue distribution and on the asymptotics of the associated orthogonal polynomials.
The main result we are going to present are the following:

- we describe the explicit derivation of the equilibrium measure for a class of potentials with discrete rotational symmetries, namely of the form

$$V(z) = |z|^{2n} - t(z^d + \bar{z}^d) \qquad n, d \in \mathbb{N}, \quad d \leq 2n \quad t > 0.$$

- We obtain the strong asymptotics for the orthogonal polynomials associated to the weight

$$e^{-NV(z)}, \quad V(z) = |z|^{2s} - t(z^s + \bar{z}^s) \qquad z \in \mathbb{C}, \ s \in \mathbb{N}, \quad t > 0,$$

  and we will show how the density of their zeroes is related to the eigenvalue distribution of the corresponding matrix model;

- We show how the conformal maps used to describe the support of the equilibrium measure for polynomial perturbation of the potential $V(z) = |z|^{2n}$ lead to a natural generalization of the concept of polynomial curves introduced in [11].

Contents

Introduction

The present work is motivated by normal matrix models, whose detailed study was initiated in the series of papers [42, 27, 23, 24]. In studying these random matrix ensembles one is interested in probability distributions of the form

$$M \mapsto \frac{1}{\mathcal{Z}_{n,N}} e^{-N\,\mathrm{Tr}(\mathcal{V}(M))} dM, \qquad \mathcal{Z}_{n,N} = \int_{\mathcal{N}_n} e^{-N\,\mathrm{Tr}(\mathcal{V}(M))} dM, \tag{1}$$

with $M \in \mathcal{N}_N$, the algebraic variety of $n \times n$ normal matrices

$$\mathcal{N}_n = \{M \ : \ [M, M^\star] = 0\} \subset \mathrm{Mat}_{n\times n}(\mathbb{C}) \ .$$

dM is the volume form induced on $\mathcal{N}_n$ which is invariant under conjugation by unitary matrices, N is a positive parameter and $\mathcal{V} : \mathbb{C} \to \mathbb{R}$ is called the *external* potential. The external potential is assumed to have sufficient growth at infinity so that the integrals in (1) are bounded. Since normal matrices are diagonalizable by unitary transformations, the probability density (1) can be reduced to the form

$$\frac{1}{Z_{n,N}} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^n \mathrm{Tr}(\mathcal{V}(\lambda_i))} dA(\lambda_1) \cdots dA(\lambda_n),$$

where λ_j are the complex eigenvalues of the normal matrix M, $dA(z)$ is the area measure, and the normalizing factor $Z_{n,N}$, called *partition function*, is given by

$$Z_{n,N} = \int_{\mathbb{C}^n} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^n (\mathcal{V}(\lambda_i))} dA(\lambda_1) \cdots dA(\lambda_n).$$

The density of eigenvalues $\rho_{n,N}(z)$ converges (in the sense of measure) in the limit

$$n \to \infty \ , \quad N \to \infty \ , \quad \frac{N}{n} \to \frac{1}{T} \ , \tag{2}$$

to the to the unique probability measure $\mu_{\mathcal{V}}$ in the plane which minimizes the functional [12, 21]

$$\mathcal{I}_{\mathcal{V}}(\mu) = \int\int \log|z-w|^{-1} d\mu(z) d\mu(w) + \frac{1}{T}\int \mathcal{V}(z) d\mu(z).$$

This functional is the *Coulomb energy functional* and for admissible potentials $\mathcal{V}(z)$[1], the existence of a unique minimizer, called *equilibrium measure*, is a well established fact [33].

[1]Following the terminology of [33], we say that $\mathcal{V} \colon \mathbb{C} \to (-\infty, \infty]$ is an *admissible potential* if it is lower semi-continuous, the set $\{z \colon \mathcal{V}(z) < \infty\}$ is of positive capacity and $\mathcal{V}(z) - \log|z| \to \infty$ as $z \to \infty$.

If $\mathcal{V}$ is C^2 and such that the Laplacian $\Delta\mathcal{V}$ is non negative, then the equilibrium measure is given by

$$d\mu_{\mathcal{V}} = \Delta V(z,\bar{z})\chi_D dA(z) \ , \tag{3}$$

where χ_D is the characteristic function of the domain D which is the support of $\mu_{\mathcal{V}}$. Furthermore,

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_{n,N} = \mathcal{I}_V(\mu_{\mathcal{V}}).$$

The asymptotic analysis of the partition function is a delicate issue, in particular if one is interested in the sub-leading terms of the expansion when the measure $d\mu_{\mathcal{V}}$ has a multiply connected support. Explicit results in this direction in the multi-component case ca be found in [7] for Hermitian matrix models.

The measure $\mu_{\mathcal{V}}$ can be also uniquely characterized by the Euler-Lagrange conditions

$$\begin{aligned} &\frac{1}{T}\mathcal{V}(z) + 2\int \log\frac{1}{|z-w|} d\mu(w) = \ell_{2D} \quad z \in \operatorname{supp}(\mu) \quad \text{quasi-everywhere} \\ &\frac{1}{T}\mathcal{V}(z) + 2\int \log\frac{1}{|z-w|} d\mu(w) \geq \ell_{2D} \quad z \in \mathbb{C} \quad \text{quasi-everywhere} \ . \end{aligned} \tag{4}$$

Here ℓ_{2D} is called the *Robin constant.* The main issue in determining the equilibrium measure is to characterize the domain D, a result which is usually achieved by exhibiting a conformal map $f : \{u \in \mathbb{C} : |u| \ \geq \ 1\} \to \mathbb{C} \setminus D$.

In the context of normal matrix models one can naturally consider associated orthogonal polynomials. In particular we can define the n-th monic orthogonal $p_n(z)$ through the following set of relations

$$\int_{\mathbb{C}} p_n(z)\overline{z}^m e^{-N\mathcal{V}(z)} dA(z) = h_{n,N}\delta_{nm}, \quad m = 0,1,2,\ldots,n \tag{5}$$

where and $h_{n,N}$ is the *norming constant.*

The statistical quantities related to the eigenvalues of a normal matrix model can be expressed in terms of the associated orthogonal polynomials $p_n(z)$ defined in (5). Some examples are given by:

- the *Christoffel–Darboux kernel*

$$K_{n,N}(z,w) = e^{-\frac{N}{2}\mathcal{V}(z)-\frac{N}{2}\mathcal{V}(w)} \sum_{k=0}^{n-1} \frac{1}{h_{j,N}} p_j(z)\overline{p_j(w)} \ ,$$

- the average density of eigenvalues $\rho_{n,N}(z)$, which can be written as

$$\rho_{n,N}(z) = \frac{1}{n} e^{-N\mathcal{V}(z)} \sum_{j=0}^{n-1} \frac{1}{h_{j,N}} |p_j(z)|^2 \ ,$$

- the partition function $Z_{n,N}$, which can be expressed as the product of the normalizing constants

$$Z_{n,N} = \prod_{j=0}^{n} h_{j,N} \ .$$

While the asymptotic density of eigenvalues can be studied using an approach from potential theory [33], the zero distribution of orthogonal polynomials remains an open issue for general potential weights despite general results in [35]. There is only a handful of potentials $\mathcal{V}(z)$ for which the polynomials $p_n(z)$ can be explicitly computed. The simplest example is the case $\mathcal{V}(z) = |z|^2$, where the orthogonal polynomials $p_n(z)$ are monomials of degree n, the constants $h_{n,N}$ and the average density of eigenvalues $\rho_{n,N}$ can be computed explicitly in terms of the Gamma function. For this choice of the potential, the matrix model is called Ginibre ensamble, [18],[19] and the density of eigenvalues $\rho_{n,N}$ converges to the area measure on the circle of radius $\sqrt{T}$. In general, for radially symmetric potentials $\mathcal{V} = \mathcal{V}(|z|)$, the orthogonal polynomials are monomials and in the limit (2), the eigenvalues distribution is supported either on a disk or an annulus by the Single-ring Theorem of Feinberg [15]. The harmonic deformation of the Gaussian case $\mathcal{V}(z) = |z|^2 +$ (harmonic) has been intensively studied. In particular the potential $\mathcal{V}(z) = |z|^2 - t(z^2 + \bar{z}^2)$ is associated to the Hermite polynomials for $|t|$ less than a critical value. In the limit (2) the distribution of eigenvalues is the area measure on an ellipse, while the distribution of the zeroes of the orthogonal polynomials is given by the (rescaled) Wigner semicircle law with support between the two foci of the ellipse [10].

The normal matrix model with a general deformation $\mathcal{V}(z) = |z|^2 + \mathrm{Re}(P(z))$ where $P(z)$ is a polynomials of fixed degree has first been considered in the seminal paper [42] where the connection with the Hele-Shaw problem and integrable structure in conformal dynamic has been pointed out. However such potential has convergence issues in the complex plane and for this reason a cut-off has been introduced in Elbau-Felder [12]. In [11] the polynomials associated to such deformation have been studied and it was argued that the Cauchy transform of the limiting zero distribution of the orthogonal polynomials is connected to the Cauchy transform of the limiting eigenvalue distribution of the matrix model. It was then conjectured that the zero distribution of the polynomials $p_n(\lambda)$ has a support on a tree like segments (the mother body) inside a domain (the droplet) that attracts the eigenvalues of the normal matrix model. For the external potential $\mathcal{V}(z) = |z|^2 + \mathrm{Re}(tz^3)$, Bleher and Kuijlaars [6] defined polynomials orthogonal with respect to a system of infinite contours on the complex plane, without any cut-off and which satisfy the same recurrence algebraic identity that is asymptotically valid for the orthogonal polynomials of Elbau and Felder. They then study the asymptotic distribution of the zeroes of such polynomials confirming Elbau's predictions. Later similar results have been obtained for the more general external potential [25] $\mathcal{V}(z) = |z|^2 - \mathrm{Re}(t|z|^k)$, $k \geq 2$ and $|t|$ sufficiently small so that the eigenvalue distribution of the matrix model has an analytic simply connected support. The case in which the eigenvalue support has singularities has been analyzed in [26] and [2]. In particular in [2] the external potential $\mathcal{V}(z) = |z|^2 - 2c \log |z - a|$, with c and a positive constants, has been studied and the strong asymptotics of the corresponding orthogonal polynomials has been derived both in the case in which the support of the eigenvalues distribution is a simply connected domain (pre-critical case) or multiply connected domain (post-critical case) and critical case. We remark that in the work [2], differently form the previous works, the zeroes of the orthogonal polynomials do not accumulate on a curve that corresponds to the mother-body (see the definition below at page 11) of the domain where the eigenvalues of the normal matrix models are distributed.

Summary of the results

In this thesis we are going to study normal random matrix models which generalize naturally the polynomially perturbed gaussian models considered for example in [12, 11, 6], focusing in particular on their eigenvalue distribution and on the asymptotics of the associated orthogonal polynomials.
The main result we are going to present are the following:

- we will describe the explicit derivation of the equilibrium measure for a class of potentials with discrete rotational symmetries, namely of the form

$$\mathcal{V}(z) = |z|^{2n} - t(z^d + \bar{z}^d) \qquad n, d \in \mathbb{N}, \quad d \leq 2n \quad t > 0.$$

  Furthermore we will show that for a certain value t_c of t a critical transition changes the support D of the equilibrium measure from a simply connected to a multiply connected domain, in particular we will be able to give an explicit description of D for $t > t_c$. There exist only few such explicit characterizations in the literature, especially in the multiply connected case.

- Using a Riemann–Hilbert approach we will obtain the strong asymptotics for the orthogonal polynomials associated to the weight

$$e^{-N\mathcal{V}(z)}, \quad \mathcal{V}(z) = |z|^{2s} - t(z^s + \bar{z}^s) \qquad z \in \mathbb{C}, \ s \in \mathbb{N}, \quad t > 0,$$

  and show that their zeroes accumulate along a curve which is related to Szegö curve. Furthermore we will show how the density of the zeroes is related to the eigenvalue distribution of the corresponding matrix model. We will also show that the same parameter t_c introduced studying the equilibrium measure determines the transition between two regimes which require different asymptotic analysis.

- We will show how the conformal maps used to describe the support of the equilibrium measure for polynomial perturbation of the potential $\mathcal{V}(z) = |z|^{2n}$ lead to a natural generalization of the concept of polynomial curves introduced in [11]. Moreover we will show that such conformal maps can be interpreted in terms of solutions of two dimensional dispersionless Toda lattice hierarchy.

Equilibrium measure

In the first chapter of this thesis we will consider potentials of the form:

$$\mathcal{V}(z) = |z|^{2n} + P(z) + \overline{P(z)} \ , \tag{6}$$

where $P(z)$ is a polynomial of degree $m \leq 2n$. Using the techniques developed in [14] we will be able to write the general form of the exterior uniformizing map $f : \{u \in \mathbb{C} : |u| > 1\} \to \mathbb{C} \setminus D$ which characterize the support D of the equilibrium measure of (6) for small polynomial perturbations. Such f can be written as

$$f(u) = ru \left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j} \right)^{\frac{1}{n}} \ , \tag{7}$$

where $m = \deg(P)$ and α_j's are constants which can be expressed in terms on the coefficients of P.

Then we will focus on the subclass of potentials of the form

$$\mathcal{V}(z) = |z|^{2n} - tz^d - t\bar{z}^d \qquad n, d \in \mathbb{N}, \quad d \leq 2n \;, \tag{8}$$

which are invariant under the group of discrete rotations of order d:

$$\mathcal{V}\left(e^{\frac{2\pi i k}{d}} z\right) = \mathcal{V}(z) \qquad k = 0, \dots, d-1 \;.$$

This symmetry will allow us to construct the conformal map in full details determining explicitly the parameters and to extend the construction developed in [14] for small values of t, to all admissible values. For such symmetric potentials it is natural to expect that the equilibrium measure of such symmetric potentials is invariant under the same group of symmetries, which motivates the following construction.

Definition 1. *For a fixed positive integer d and a Borel probability measure μ the associated d*-fold rotated measure *$\mu^{(d)}$ is defined to be*

$$\mu^{(d)} = \frac{1}{d} \sum_{k=0}^{d-1} \mu_k^{(d)} \;,$$

where the kth summand is given by

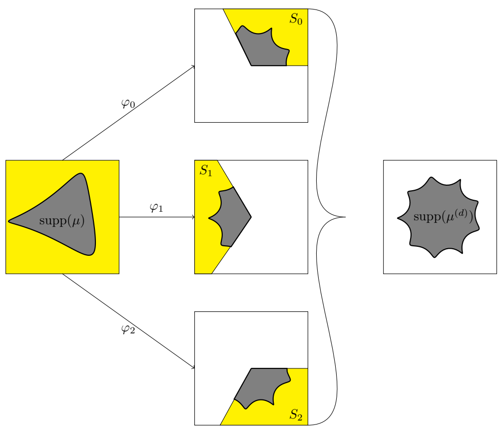
$$\mu_k^{(d)}(B) = \mu\left(\varphi_k^{-1}(B \cap S_k)\right)$$

for any Borel set $B \subseteq \mathbb{C}$, where

$$S_k = \left\{ z \in \mathbb{C} \;:\; \frac{2\pi k}{d} \leq \arg(z) < \frac{2\pi(k+1)}{d} \right\}$$

for $k = 0, \dots, d-1$, and

$$\varphi_k \colon \mathbb{C} \to S_k \;, \quad \varphi_k(re^{i\theta}) = r^{\frac{1}{d}} e^{\frac{i\theta}{d}} e^{\frac{2\pi i k}{d}} \;.$$



Lemma 1. *If the admissible potential $\mathcal{V}(z)$ can be written in terms of another admissible potential $Q(z)$ as*

$$\mathcal{V}(z) = \frac{1}{d} Q(z^d)$$

for some positive integer d, then the equilibrium measure for $\mathcal{V}$ is given by the d-fold rotated equilibrium measure of Q, i.e.,

$$\mu_{\mathcal{V}} = \mu_Q^{(d)} \;.$$

As a corollary, the problem of finding the equilibrium measure for the class (8) reduces to the study of the following family of potentials:

$$Q(z) = d\left(|z|^{\frac{2n}{d}} - tz - t\bar{z}\right) \;.$$

The major advantage in dealing with Q instead of $\mathcal{V}$ is that the support of its equilibrium measure is always simply connected and thus an exterior uniformizing map always exist.

By differentiating (4) it is easy to find the density of μ_Q with respect to the Lebesgue measure dA in the complex plane as

$$d\mu_Q(z) = \frac{1}{4\pi T}\Delta Q(z)\chi_K(z)dA(z) = \frac{n^2}{\pi T d}|z|^{\frac{2n}{d}-2}\chi_K(z)dA(z) ,$$

where $K = \mathrm{supp}(\mu_Q)$ and χ_K stands for the characteristic function of K. The support set K will be expressed in terms of its exterior uniformizing map

$$f\colon \{u \in \mathbb{C}\colon\ |u| > 1\} \to \mathbb{C} \setminus K$$

analytic and univalent for $|u| > 1$ and fixed by the asymptotic behaviour

$$f(\infty) = \infty , \quad f(u) = ru\left(1 + \mathcal{O}\left(\frac{1}{u}\right)\right) \quad u \to \infty ,$$

where r is real and positive, called the *conformal radius* of K.

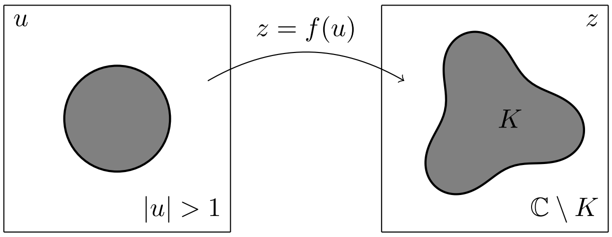


Figure 1: Illustration of the exterior conformal map

The results concerning the equilibrium measure for Q can be summarized in the following

Theorem 1. *Let $0 < d < n$ or $n < d < 2n$, and define the critical value of the parameter t as*

$$t_c = \frac{n}{d}\left(\frac{T}{2n-d}\right)^{\frac{2n-d}{2n}} .$$

The equilibrium measure for the potential

$$Q(z) = d\left(|z|^{\frac{2n}{d}} - tz - t\bar{z}\right)$$

is absolutely continuous with respect to the area measure with density

$$d\mu_Q(z) = \frac{n^2}{\pi T d}|z|^{\frac{2n}{d}-2}\chi_K(z)dA(z) ,$$

and its support K is simply connected. The exterior uniformizing map of K is of the form

$$f(u) = \begin{cases} ru\left(1 - \dfrac{\alpha}{u}\right)^{\frac{d}{n}} & t \le t_c \\[2ex] r\left(u - \dfrac{1}{\bar{\alpha}}\right)\left(1 - \dfrac{\alpha}{u}\right)^{\frac{d}{n}-1} & t > t_c , \end{cases}$$

where $r = r(t)$ and $\alpha = \alpha(t)$ are given as follows:

- ***Pre-critical case*** $0 < t < t_c$ ***:***
  The radius $r = r(t)$ is a particular solution of the equation

$$r^{\frac{4n}{d}-2} - \frac{T}{n} r^{\frac{2n}{d}-2} + \frac{n-d}{n}\frac{d^2}{n^2} t^2 = 0 \ . \tag{9}$$

  1. *For $0 < d < n$, (9) has two distinct positive solutions $0 < r_-(t) < r_+(t) < \left(\frac{T}{n}\right)^{\frac{d}{2n}}$ such that $r_-(0) = 0$ and $r_+(0) = \left(\frac{T}{n}\right)^{\frac{d}{2n}}$ and $r_-(t_c) = r_+(t_c)$. The conformal map corresponds to the choice $r = r_+(t)$.*
  2. *For $n < d < 2n$, (9) has a unique positive solution $r = r(t)$.*

  The value of the parameter α is given in terms of r as

$$\alpha = -\frac{d}{n} t\, r^{-\frac{2n-d}{d}} \ .$$

- ***Critical case*** $t = t_c$ ***:***
  The parameters r and α are given explicitly by

$$r = r_c = \left(\frac{T}{2n-d}\right)^{\frac{d}{2n}} \qquad \text{and} \qquad \alpha = \alpha_c = -1 \ .$$

- ***Post-critical case*** $t > t_c$ ***:***
  The parameters r and α are given explicitly by

$$r = \left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{d}{n} t\right)^{\frac{d-n}{2n-d}} \qquad \text{and} \qquad \alpha = -\left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{dt}{n}\right)^{-\frac{n}{2n-d}} \ .$$

A straightforward application of Lemma 1 to the potential Q in Theorem 1 yields easily to the analog results for the potential $\mathcal{V}$:

Corollary 1. *Let $0 < d < n$ or $n < d < 2n$. The equilibrium measure for the potential*

$$\mathcal{V}(z) = |z|^{2n} - t z^d - t \bar{z}^d$$

is given by $\mu_Q^{(d)}$, the d-fold rotated equilibrium measure of Q given above.

More precisely, $\mu_{\mathcal{V}}$ is absolutely continuous with respect to the area measure with density

$$d\mu_{\mathcal{V}}(z) = \frac{n^2}{\pi T} |z|^{2n-2} \chi_D(z) dA(z) \ ,$$

and the support set D of $\mu_{\mathcal{V}}$ can be described as follows:

- *For $t \leq t_c$, D is simply connected and it is given by the exterior conformal map*

$$g(u) = \left(f(u^d)\right)^{\frac{1}{d}} = r^{\frac{1}{d}} u \left(1 - \frac{\alpha}{u^d}\right)^{\frac{1}{n}} \ ,$$

  where r and α are given in Theorem 1.

- *For $t > t_c$, D has d disjoint simply connected components and their boundary is parametrized by*

$$g(u) = r^{\frac{1}{d}} u \left(1 - \frac{1}{\bar{\alpha} u^d}\right)^{\frac{1}{d}} \left(1 - \frac{\alpha}{u^d}\right)^{\frac{1}{n} - \frac{1}{d}} ,$$

where u lies on the unit circle and the factor

$$\left(1 - \frac{1}{\bar{\alpha} u^d}\right)^{\frac{1}{d}} \sim 1 + \mathcal{O}\left(\frac{1}{u}\right) \quad u \to \infty$$

is defined with some suitably defined branch cuts on $|u| \geq 1$, with r and α given in Theorem 1.

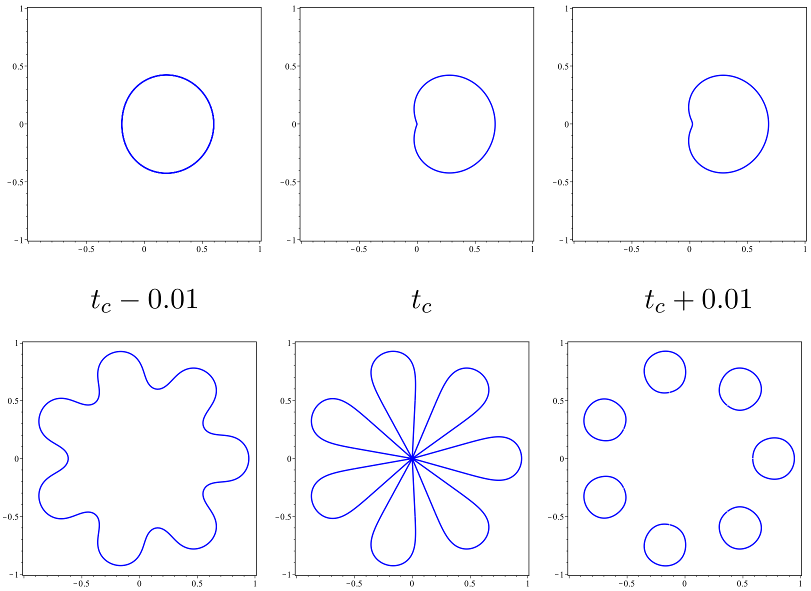


Figure 2: The equilibrium supports for $n = 9$, $d = 7$, $T = 1$ (above for Q and below for V)

Remark. *The critical transition which characterize the domain D for $t = t_c$ (see picture above) is highly non generic and it might be studied with the techniques developed in [5].*

Orthogonal polynomials

In the second chapter we will restrict ourself to consider potentials (8) with the choice $d = n = s$, in particular we will study the strong asymptotic of the polynomials $p_n(z)$ orthogonal with respect to the external potential of the form

$$e^{-N\mathcal{V}(z)}, \quad \mathcal{V}(z) = |z|^{2s} - t(z^s + \bar{z}^s) \qquad z \in \mathbb{C},\ s \in \mathbb{N}, \quad t > 0. \tag{10}$$

where the potential $\mathcal{V}(z)$ has an s-discrete rotational symmetry.

The polynomials $p_n(z)$ defined in (5) orthogonal with respect to the weight (10) show a different behaviour in the two cases which also appeared in 1, i.e.

- pre-critical: $t < t_c$;
- post-critical $t > t_c$.

The $\mathbb{Z}_s$-symmetry of the orthogonality measure (10) is inherited by the corresponding orthogonal polynomials. Indeed the non-trivial orthogonality relations are

$$\int_{\mathbb{C}} p_n(z)\bar{z}^{js+l} e^{-N\mathcal{V}(z)} dA(z), \quad j = 0, \dots, k-1,$$

where k and l are such that

$$n = ks + l, \qquad 0 \le l \le s-1,$$

i.e., the n-th monic orthogonal polynomial satisfies the relation

$$p_n(e^{\frac{2\pi i}{s}} z) = e^{\frac{2\pi i n}{s}} p_n(z).$$

It follows that there exists a monic polynomial $q_k^{(l)}$ of degree k such that

$$p_n(z) = z^l q_k^{(l)}(z^s).$$

Therefore the sequence of orthogonal polynomials $\{p_n(z)\}_{n=0}^{\infty}$ can be split into s subsequences labelled by the remainder $l \equiv n \mod s$, and the asymptotics along the different subsequences can be studied via the sequences of reduced polynomials

$$\left\{ q_k^{(l)}(u) \right\}_{k=0}^{\infty}, \qquad l = 0, 1, \dots, s-1.$$

By a simple change of coordinates it is easy to see that the monic polynomials in the sequence $\{q_k^{(l)}\}_{k=0}^{\infty}$ are orthogonal with respect to the measure

$$|u|^{-2\gamma} e^{-N(|u|^2 - tu - t\bar{u})} dA(u), \quad \gamma := \frac{s-l-1}{s} \in [0,1), \tag{11}$$

namely they satisfy the orthogonality relations

$$\int_{\mathbb{C}} q_k^{(l)}(u)\bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)} dA(u) = 0, \qquad j = 0, \dots, k-1.$$

As a result of this symmetry reduction, starting from the class of measures (10), it is sufficient to consider the orthogonal polynomials with respect to the family of measures (11).

Remark. *It is clear from the above relation that for $l = s-1$ one has $\gamma = 0$ and the polynomials $q_k^{(s-1)}(u)$ are monomials in the variable $(u - t)$, that is,*

$$q_k^{(s-1)}(u) = (u-t)^k.$$

It follows that the monic polynomials $p_{ks+s-1}(z)$ have the form

$$p_{ks+s-1}(z) = z^{s-1}(z^s - t)^k.$$

Define

$$z_0 = \frac{t_c^2}{t^2}$$

and consider the level curve $\hat{\Gamma}_r$

$$\hat{\Gamma}_r := \left\{ z \in \mathbb{C},\ \mathrm{Re}\ \hat{\phi}_r(z) = 0,\ \ |z^s - t| \le z_0 t \right\}, \tag{12}$$

where r is a positive constant and

$$\hat{\phi}_r(z) = \log(t - z^s) + \frac{z^s}{tz_0} - \log rt + \frac{r-1}{z_0}. \tag{13}$$

We consider the usual counter-clockwise orientation for $\hat{\Gamma}_r$. These level curves consist of s closed contours contained in the set D, where

$$D := \{z \in \mathbb{C},\ |z^s - t| \le t_c\}. \tag{14}$$

Define the measure $\hat{\nu}$ associated with this family of curves given by

$$d\hat{\nu} = \frac{1}{2\pi i s} d\hat{\phi}_r(z), \tag{15}$$

and supported on $\hat{\Gamma}_r$.

We will be able to prove the following results:

Lemma 2. *The a-priori complex measure $d\hat{\nu}$ in* (15) *is a probability measure on the contour $\hat{\Gamma}_r$ defined in* (12) *for $0 < r \le \frac{t}{t_c}$.*

Theorem 2. *The zeroes of the polynomials $p_n(z)$ defined in* (5) *behave as follows*

- *for $n = sk+s-1$ let $\omega = e^{\frac{2\pi i}{s}}$. Then $t^{\frac{1}{s}}, \omega t^{\frac{1}{s}}, \ldots, \omega^{k-1} t^{\frac{1}{s}}$ are zeroes of the polynomials p_{ks+s-1} with multipicity k and $z = 0$ is a zero with multiplicity $s-1$.*

- *for $n = ks + l$, $l = 0, \ldots, s-2$ the polynomial $p_n(z)$ has a zero in $z = 0$ with multiplicity l and the remaining zeroes in the limit $n, N \to \infty$ such that*

  $$N = \frac{n-l}{T}$$

  accumulates on the level curves $\hat{\Gamma}_r$ as in (12) *with $r = 1$ for $t < t_c$ and $r = \frac{t_c^2}{t^2}$ for $t > t_c$. Namely the curve $\hat{\Gamma}$ on which the zeroes accumulate is given by*

  $$\hat{\Gamma}: \quad \left| (t - z^s) \exp\left(\frac{z^s t}{t_c^2} \right) \right| = \begin{cases} t & \textit{pre-critical case } 0 < t < t_c, \\ \frac{t_c^2}{t} e^{\frac{t^2}{t_c^2} - 1} & \textit{post-critical case } t > t_c, \end{cases} \tag{16}$$

  with $|z^s - t| \le z_0 t$. The measure $\hat{\nu}$ in (15) *is the weak-star limit of the normalized zero counting measure ν_n of the polynomials p_n for $n = sk + l$, $l = 0, \ldots, s-2$.*

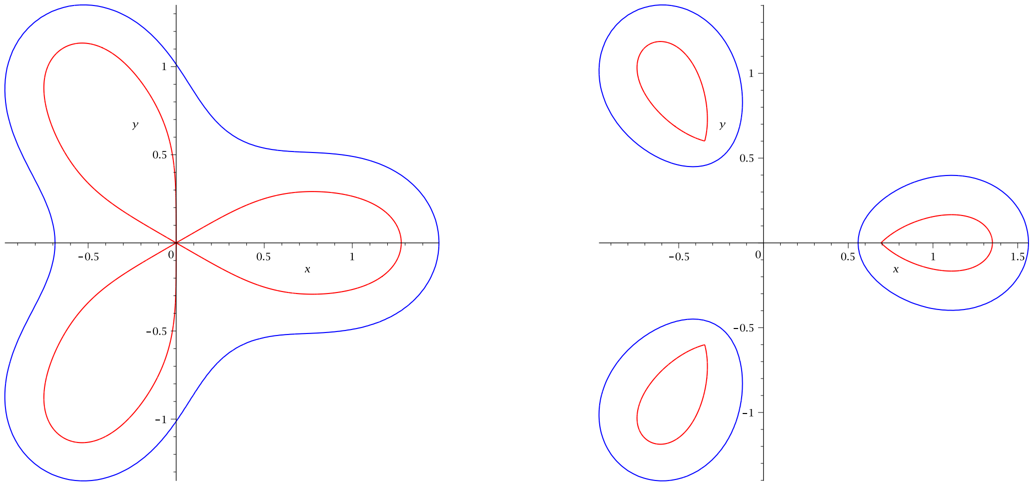


Figure 3: The blue contour is the boundary of the support D defined in (14) and the red contour is $\hat{\Gamma}$ defined in (16). Here $s = 3$ and $t < t_c$ (left figure) and $t > t_c$ (right figure). In the post-critical case, the points where the curve $\hat{\Gamma}$ is not smooth are the solutions of the equation $z^s = t(1 - z_0)$, with $z_0 = t_c^2/t^2$.

From Theorem 2 the following identity follows immediately.

Lemma 3. *The equilibrium measure $\mu_{\mathcal{V}}$ in* (3) *of the eigenvalue distribution of the normal matrix model and the measure $\hat{\nu}$ in* (15) *of the zero distribution of the orthogonal polynomials are related by*

$$\int_D \frac{d\mu_{\mathcal{V}}(w)}{z-w} = \int_{\hat{\Gamma}} \frac{d\hat{\nu}(w)}{z-w}, \quad z \in \mathbb{C} \backslash D. \tag{17}$$

Remark. *The identity* (17) *in Lemma 3 is expected to hold in general for a large class of normal matrix models. It has been verified for several other potentials (see, for example [1, 11, 2, 6, 25]), but a proof of such result for a generic potential is still missing.*

We also observe that for the orthogonal polynomials appearing in random matrices, in some cases, the asymptotic distribution of the zeroes is supported on the so-called mother body or potential theoretic skeleton of the support D of the eigenvalue distribution. We recall that a measure ν is a mother body *for a domain D with respect to a measure μ if [20]*

1) $\displaystyle\int \log|z-w|d\mu(w) \leq \int \log|z-w|d\nu(w)$, *for $z \in \mathbb{C}$ with equality for z outside D;*

2) $\nu \geq 0$*;*

3) $|\operatorname{supp}\nu| = 0$*;*

4) the support of ν does not disconnect any part of D from $\bar{D}^c$.

The problem of constructing mother bodies is not always solvable and the solution is not always unique [34].

Concerning the explicit examples appearing in the random matrix literature, for the exponential weight $V(z) = |z|^2 + \mathrm{Re}(P(z))$ where $P(z)$ is polynomial, the support of the *zero distribution of the orthogonal polynomials is indeed the mother body of the domain that corresponds to the eigenvalue distribution of the matrix model (see, e.g., [6, 25, 18, 12, 42]). While for the case studied by Bertola et al. [2] and the present case, the support of the zero distribution of the orthogonal polynomials does not have property 4) and therefore it is not a mother body of the set D.*

In order to describe the pointwise asymptotics of the polynomials $p_n(z)$ in the pre-critical case ($t < t_c$) and post-critical case ($t > t_c$) we introduce the function

$$\hat{\phi}(z) = \begin{cases} \hat{\phi}_{r=z_0}(z) & 0 < t < t_c \quad \text{(pre-critical)} \\ \\ \hat{\phi}_{r=1}(z) & t_c < t \qquad\quad \text{(post-critical)}, \end{cases} \tag{18}$$

where $\hat{\phi}_r(z)$ is given by (13).

Theorem 3 (Pre-critical case). *For $0 < t < t_c$ the polynomial $p_n(z)$ with $n = ks + l$, $l = 0, \dots, s-2$, $\gamma = \frac{s-l-1}{s} \in (0,1)$, have the following asymptotic behaviour when $n, N \to \infty$ in such a way that $NT = n - l$:*

(1) for z in compact subsets of the exterior of $\hat{\Gamma}$ one has

$$p_n(z) = z^{s-1}(z^s - t)^{k-\gamma}\left(1 + O\left(\frac{1}{k^{2+\gamma}}\right)\right);$$

(2) for z near $\hat{\Gamma}$ and away from $z = 0$,

$$p_n(z) = z^{s-1}(z^s - t)^{k-\gamma}\left[1 + \frac{e^{-k\hat{\phi}(z)}}{k^{1+\gamma}}\left(\frac{1}{\Gamma(-\gamma)}\left(1 - \frac{1}{z_0}\right)^{-1-\gamma}\frac{t}{z^s}\left(1 - \frac{t}{z^s}\right)^{\gamma} + O\left(\frac{1}{k}\right)\right)\right], \tag{19}$$

where $\hat{\phi}(z)$ has been defined in (18);

(3) for z in compact subsets of the interior of $\hat{\Gamma}$ and away from $z = 0$,

$$p_n(z) = z^l \frac{e^{-\frac{kz^s}{tz_0}}}{k^{1+\gamma}}\left(\frac{(-t)^{k+1}}{\Gamma(-\gamma)}\frac{1}{z^s}\left(1 - \frac{1}{z_0}\right)^{-1-\gamma} + O\left(\frac{1}{k}\right)\right);$$

(4) for z in a neighbourhood of $z = 0$, we introduce the function $\hat{w}(z) = \hat{\phi}(z) + 2\pi i$ if $z^s \in \mathbb{C}_-$ and $\hat{w}(z) = \hat{\phi}(z)$ if $z^s \in \mathbb{C}_+$. Then

$$p_n(z) = z^l(z^s - t)^{k-\gamma}\left(\frac{z^s}{\hat{w}(z)}\right)^{\gamma}\left[(\hat{w}(z))^{\gamma} - \frac{e^{-k\hat{\phi}(z)}}{k^{\gamma}}\left(\tilde{\Psi}_{12}(k\hat{w}(z)) + O\left(\frac{1}{k}\right)\right)\right],$$

where the $(1,2)$-entry of the matrix

$$\tilde{\Psi}(\xi) = \begin{pmatrix} 1 & -\frac{1 - e^{-2\gamma\pi i}}{2\pi i}\int_{\mathbb{R}^-}\frac{(\zeta^{\gamma})_+ e^{\zeta} d\zeta}{\zeta - \xi} \\ 0 & 1 \end{pmatrix}.$$

We observe that in compact subsets of the exterior of $\hat{\Gamma}$ there are no zeroes of the polynomials $p_n(z)$. The only possible zeroes are located in $z = 0$ and in the region where the second term in parenthesis in the expression (19) is of order one. Since $\operatorname{Re}\hat{\phi}(z)$ is positive inside $\hat{\Gamma}$ and negative outside $\hat{\Gamma}$ it follows that the possible zeroes of $p_n(z)$ lie inside $\hat{\Gamma}$ and are determined by the condition

$$\log|\hat{\phi}(z)| = -(1+\gamma)\frac{\log k}{k} + \frac{1}{k}\log\left(\frac{1}{|\Gamma(-\gamma)|}\frac{t}{|z|^s}\left|1-\frac{1}{z_0}\right|^{-1-\gamma}\left|1-\frac{t}{z^s}\right|^{\gamma}\right), \quad |z^s - t| \le t. \tag{20}$$

The above expression shows that the zeroes of the polynomials $p_n(z)$ are within a distance $O(1/k)$ from the level curve (20). Such curve converges to $\hat{\Gamma}$ defined in (16) at a rate $O(\log k/k)$.

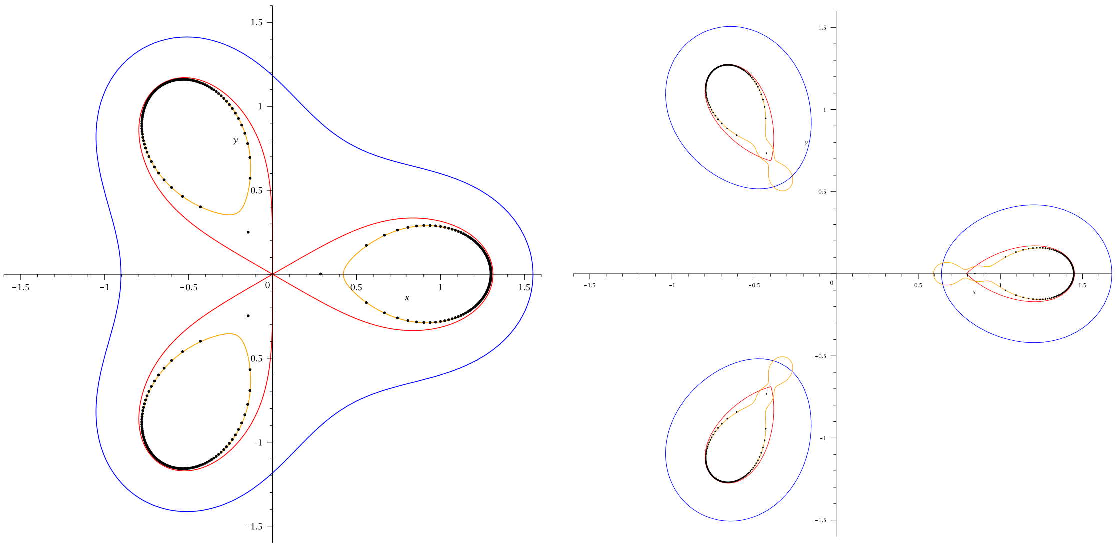


Figure 4: The blue contour is the boundary of the domain D defined in (14) and the red contour is $\hat{\Gamma}$ defined in (16), the yellow contour is given by (20) in the pre-critical case and (22) in the post-critical case. The dots are the zeroes of the polynomial $p_n(z)$ for $s = 3$, $n = 285$, $l = 0$ and $t < t_c$ on the right and $t > t_c$ on the left, respectively.

Theorem 4 (Post-critical case). *For $t > t_c$ the polynomials $p_n(z)$ with $n = ks + l$, $l = 0, \ldots, s-2$, and $\gamma = \frac{s-l-1}{s} \in (0,1)$ have the following behaviour when $n, N \to \infty$ in such a way that $NT = n - l$*

(1) for z in compact subsets of the exterior of $\hat{\Gamma}$ one has

$$p_n(z) = z^l(z^s - t)^{k-\gamma}(z^s + t(z_0 - 1))^{\gamma}\left(1 + O\left(\frac{1}{k}\right)\right),$$

(2) for z in the region near $\hat{\Gamma}$ and away from the points $z^s = t(1 - z_0)$ one has

$$p_n(z) = z^l(z^s - t)^{k-\gamma}(z^s + t(z_0 - 1))^{\gamma}\left(1 - \frac{e^{-k\hat{\phi}(z)}}{k^{\frac{1}{2}+\gamma}}\frac{\gamma z_0^2}{c}\frac{t((z^s-t)z^s)^{\gamma}}{(z^s + t(z_0-1))^{2\gamma+1}} + O\left(\frac{1}{k}\right)\right), \tag{21}$$

where $\hat{\phi}(z)$ *has been defined in* (18) *and*

$$c = -\frac{\Gamma(1-\gamma)}{\sqrt{2\pi}} \left(\frac{1-z_0}{z_0}\right)^{\gamma} z_0 \ ;$$

(3) for z *in compact subsets of the interior region of* $\hat{\Gamma}$ *one has*

$$p_n(z) = z^l \frac{e^{k\frac{t-z^s}{tz_0}}}{k^{\frac{1}{2}+\gamma}} \frac{t\gamma z_0^2}{c} \left(\frac{tz_0}{e}\right)^k \left(\frac{z^{s\gamma}}{(z^s + t(z_0-1))^{\gamma+1}} + O\left(\frac{1}{k}\right)\right) \ ;$$

(4) in the neighbourhood of each of the points that solve the equation $z^s = t(1-z_0)$ *one has*

$$p_n(z) = z^l (z^s - t)^{k-\gamma} \left(\frac{z^s + t(z_0-1)}{\sqrt{k}\hat{w}(z)}\right)^{\gamma} e^{-k\hat{\phi}(z)} \left(\mathcal{U}(-\gamma - \frac{1}{2}; \sqrt{2k}\hat{w}(z)) + O\left(\frac{1}{k}\right)\right);$$

where $\mathcal{U}(a;\xi)$ *is the parabolic cylinder function satisfying the equation*

$$\frac{d^2}{d\xi^2}\mathcal{U} = \left(\frac{1}{4}\xi^2 + a\right)\mathcal{U}$$

and $\hat{w}^2(z) = -\hat{\phi}(z) - 2\pi i$ *for* $z^s \in \mathbb{C}_-$ *and* $\hat{w}^2(z) = -\hat{\phi}(z)$ *for* $z^s \in \mathbb{C}_+$.

We observe that in compact subsets of the exterior of $\hat{\Gamma}$ the polynomials $p_n(z)$ have zero at $z = 0$ with multiplicity l. The other possible zeroes are located in the region where the second term in parenthesis in the expression (21) is of order one. Since $\operatorname{Re}\hat{\phi}(z)$ is positive inside $\hat{\Gamma}$, this is possible and it follows that

$$\log|\hat{\phi}(z)| = -(1+\gamma)\frac{\log k}{k} + \frac{1}{k}\log\left(\frac{t\gamma z_0^2}{|c|}\frac{|(z^s - t)z^s|^{\gamma}}{|z^s + t(z_0-1)|^{2\gamma+1}}\right). \qquad (22)$$

The above expression shows that the zeroes of the polynomials $p_n(z)$ are within a distance $O(1/k)$ from the level curve (22). Such curve converges to $\hat{\Gamma}$ defined in (16) at a rate $O(\log k/k)$.

We will prove Theorem 3 and Theorem 4 by reducing the planar orthogonality relations of the polynomials $p_n(z)$ to orthogonality relations with respect to a complex density on a contour. More precisely, the sequence of polynomials $p_n(z)$ can be reduced to s families of polynomials $q^{(l)}(z^s)$, $l = 0, \dots, s-1$ and the orthogonality relations of the polynomials $q^{(l)}(u)$ can be reduced to orthogonality relations on a contour. We then reformulate such orthogonality relation as a Riemann-Hilbert problem. We perform the asymptotic analysis of the polynomials $p_n(z)$ using the nonlinear steepest descent/stationary phase method introduced by Deift and Zhou [9].

The zeroes of $p_n(z)$ accumulate along an open contour as shown in Figure 3. The determination of this contour will be the first step in the analysis.

The Riemann-Hilbert method will give strong and uniform asymptotics of the polynomials $p_n(z)$ in the whole complex plane. The asymptotic behaviour of the polynomials $p_n(z)$ in the leading and sub-leading order can be expressed in terms of elementary functions in the pre-critical case $t < t_c$, while in the post-critical case $t > t_c$, in some regions of the complex plane, in terms of parabolic cylinder functions as in [9],[22]. The proof of Theorem 2 can be deduced from the strong asymptotic of the orthogonal polynomials.

Integrable hierarchies

In the third chapter we will investigate the connection between conformal maps and integrable hierarchies. In [42, 43] Wiegmann and Zabrodin realized that conformal maps from the exterior of the unit disc to the exterior of a simple closed analytic curve admit as functions of the exterior harmonic moments $(t_k)_{k=1}^{\infty}$ of the analytic curve the structure of an integrable hierarchy: the dispersionless Toda lattice hierarchy.

In [11] a certain class of curves is introduced and it is shown how they are connected with the dispersionless Toda lattice hierarchy. The important property of those curves is that they are uniquely characterized by their exterior harmonic moments, which play the role of the times of hierarchy. In the following we will introduce a generalization of such curves which naturally arise as boundary of supports of equilibrium measures of normal matrix models: these curves exhibits similar properties to the ones in [11] and allow a very similar analysis.

A rigorous treatment of the connection between conformal maps and Toda lattice hierarchy in full generality is given in [38, 39], however the class of maps we are interested in allows a particularly explicit analysis, which we think is worth to mention.

Consider potentials of the form

$$\mathcal{V}(z) = |z|^{2n} + P(z) + \overline{P(z)} \ ,$$

with $P(z)$ a polynomial and $\deg(P) = m < 2n$, whose associated equilibrium measure can be expressed as

$$d\mu(z) = \frac{1}{\pi T}\rho(z\overline{z})dA(z) = \frac{n^2}{\pi T}|z|^{2n-2}\chi_D(z)dA(z) \ .$$

The conformal map characterizing the support D is given by

$$f(u) = ru\left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right)^{\frac{1}{n}} \ , \tag{23}$$

for some constants α_j $j = 1, \ldots, m$ and hence its $n-$th power is a rational function, i.e.

$$(f(u))^n = r^n u^n \left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right) \ .$$

This property of the conformal map makes possible to generalize the concept of *polynomial curve* introduced by Elbau in [11] to describe the class of curves which bound the support of the equilibrium measure for potentials of the form

$$|z|^2 + P(z) + \overline{P(z)} \ .$$

Definition 2. *An $(n, m)-$*polynomial curve *is a smooth simple closed curve in the complex plane with a parametrization $f : \{u \in \mathbb{C} : |u| = 1\} \to \mathbb{C}$ of the form*

$$f(u) = ru\left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right)^{\frac{1}{n}}$$

with $r > 0$ and m finite. The standard (counterclockwise) orientation of the unit circle in $\mathbb{C}$ induces an orientation on the curve. We say that a polynomial curve is positively oriented *if this orientation is counterclockwise.*

Definition 3. *The* exterior harmonic moments $(t_k)_{k=1}^{\infty}$ *of a $(n, m)-$polynomial curve γ*

$$t_k := \frac{1}{2\pi i k} \oint_\gamma z^{-k} \bar{z} \rho(z\bar{z}) dz \ ,$$

where

$$\rho(z\bar{z}) = n^2 |z|^{2n-2} \ .$$

The interior harmonic moments $(v_k)_{k=1}^{\infty}$ *of γ are*

$$v_k := \frac{1}{2\pi i} \oint_\gamma z^k \bar{z} \rho(z\bar{z}) dz \ .$$

It is also useful to introduce the *total mass* t_0:

$$t_0 = r^{2n} \frac{1}{2\pi i} \oint_\gamma \bar{z}^n dz^n = n r^{2n} \left(1 + \sum_{j=1}^{m} \frac{n-j}{n} |\alpha_j|^2 \right) \ .$$

We will prove that exterior harmonic moments together with the total mass determine uniquely an $(n, m)-$polynomial curve:

Theorem 5. *Given any $(t_3, \ldots, t_m) \in \mathbb{C}^{m-2}$ and $n \in \mathbb{N}$, there exist $\delta_0, \delta_1, \delta_2 > 0$ s.t. for all $0 < t_0 < \delta_0$, $|t_1| < \delta_1$ and $|t_2| < \delta_2$, there exists a unique positively oriented $(n, m)-$polynomial curve encircling the origin with total mass t_0 and exterior harmonic moments $(t_k)_{k=1}^m$ with $t_k = 0$ for $k > m$.*

In order to show the connection between conformal maps and integrable hierarchies it is useful to introduce the *Schwarz function* associated to a curve.

Definition 4. *The* Schwarz function *(see e.g. [8]) of a nonsingular analytic Jordan curve γ is defined as the analytic continuation (in a neighborhood of γ) of the function $S(z) = \bar{z}$ on γ.*

Let f be a parametrization of γ: using the definition of the Schwarz function S we have that in a neighborhood of the unit circle

$$S(f(u)) = \overline{f} \left(\frac{1}{u} \right) \ ,$$

Hence it follows that in a neighborhood of γ we have

$$S(z) = \overline{f} \left(\frac{1}{f^{-1}(z)} \right) \ .$$

Considering an $(n, m)-$polynomial curve γ it is a trivial fact that the harmonic moments of γ can be written in terms of the $n-$th power of the Schwarz function. Indeed since on γ we have $\bar{z} = S(z)$ we can write

$$t_0 = n^2 r^{2n} \frac{1}{2\pi i} \oint_\gamma z^{n-1} S^n(z) dz$$
$$t_k = n^2 r^{2n} \frac{1}{2\pi i k} \oint_\gamma z^{n-1-k} S^n(z) dz$$
$$v_k = n^2 r^{2n} \frac{1}{2\pi i} \oint_\gamma z^{n-1+k} S^n(z) dz \ .$$

Redefining

$$t_k := \frac{1}{n^2} r^{-2n} t_k \qquad\qquad v_k := \frac{1}{n^2} r^{-2n} v_k$$

we can write the Laurent expansion of $S^n(z)$ for $z \to \infty$ in terms of the harmonic moments

$$S^n(z) = \sum_{k=1}^m k t_k z^{k-n} + \frac{t_0}{z^n} + \sum_{k=1}^\infty v_k z^{-k-n} = z^{-n+1} \left[\sum_{k=1}^m k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^\infty v_k z^{-k-1} \right]$$

Toda lattice hierarchy The two dimensional *Toda lattice equation* is

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} b(n) = e^{b(n+1)-b(n)} - e^{b(n)-b(n-1)} \ , \tag{24}$$

where b is a function of the continuous variables t_1 and $\tilde{t}_1$ and of the discrete variable $n \in \mathbb{N}$. In order to obtain the *dispersionless limit* of (24) it is useful to define

$$a(n\varepsilon, t_1, \tilde{t}_1) := b(n) - b(n-1) \ ,$$

and $t_0 := n\varepsilon$, so that t_0 becomes a continuous variable in the continuum limit $\varepsilon \to 0$, $n \to \infty$. By rescaling $t_1 \to t_1/\varepsilon$ and $\tilde{t}_1 \to \tilde{t}_1/\varepsilon$ equation (24) reduces to

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} a(n\varepsilon, t_1, \tilde{t}_1) = \frac{1}{\varepsilon^2} \left(e^{a((n+1)\varepsilon, t_1, \tilde{t}_1)} - 2 e^{a(n\varepsilon, t_1, \tilde{t}_1)} + e^{a((n-1)\varepsilon, t_1, \tilde{t}_1)} \right) \ ,$$

which, taking the formal limit $\varepsilon \to 0$, $n \to \infty$, yields to the *dispersionless Toda equation*

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} a(t_0, t_1, \tilde{t}_1) = \frac{\partial^2}{\partial t_0^2} e^{a(t_0, t_1, \tilde{t}_1)} \ . \tag{25}$$

Equation (24) can be seen as an element of the so called *Toda lattice hierarchy*, introduced by Ueno and Takasaki in [41]. A review of the Toda lattice hierarchy and its dispersionless limit can be found in [37].

Definition 5 ([37])**.** *The Toda lattice hierarchy is defined by the Lax-Sato equations*

$$\frac{\partial L}{\partial t_k} = \frac{1}{\varepsilon}[M^{(k)}, L], \qquad \frac{\partial \tilde{L}}{\partial t_k} = \frac{1}{\varepsilon}[M^{(k)}, \tilde{L}], \tag{26}$$
$$\frac{\partial L}{\partial \tilde{t}_k} = \frac{1}{\varepsilon}[L, \tilde{M}^{(k)}], \qquad \frac{\partial \tilde{L}}{\partial \tilde{t}_k} = \frac{1}{\varepsilon}[\tilde{L}, \tilde{M}^{(k)}], \tag{27}$$

$k \in \mathbb{N}$*, for the difference operators*

$$L(t_0, \underline{t}, \underline{\tilde{t}}) = r(t_0, \underline{t}, \underline{\tilde{t}}) e^{\varepsilon \partial_{t_0}} + \sum_{j=0}^\infty a_j(t_0, \underline{t}, \underline{\tilde{t}}) e^{-j\varepsilon \partial_{t_0}} \quad \textit{and}$$
$$\tilde{L}(t_0, \underline{t}, \underline{\tilde{t}}) = \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}}) e^{-\varepsilon \partial_{t_0}} + \sum_{j=0}^\infty \tilde{a}_j(t_0, \underline{t}, \underline{\tilde{t}}) e^{j\varepsilon \partial_{t_0}},$$

$\underline{t} = (t_k)_{k=1}^{\infty}$, $\underline{\tilde{t}} = (\tilde{t}_k)_{k=1}^{\infty}$, *on the space of all analytic functions in* t_0, *where*

$$M^{(k)} = (L^k)_+ + \frac{1}{2}(L^k)_0 \quad \textit{and} \quad \tilde{M}^{(k)} = (\tilde{L}^k)_- + \frac{1}{2}(\tilde{L}^k)_0, \quad k \in \mathbb{N}. \tag{28}$$

For any operator the subscripts $+$, 0, *and* $-$ *denote its positive, constant and negative part in the* shift operator

$$e^{\varepsilon\partial_{t_0}} := \sum_{j=0}^{\infty} \frac{1}{j!}\varepsilon^j \frac{\partial^j}{\partial t_0^j}.$$

Proposition 1 ([37])**.** *Let* L *and* $\tilde{L}$ *be a solution of the Toda lattice hierarchy. Then, with* $M^{(k)}$ *and* $\tilde{M}^{(k)}$ *given by* (28), *the compatibility relations*

$$\frac{\partial M^{(j)}}{\partial t_k} - \frac{\partial M^{(k)}}{\partial t_j} = \frac{1}{\varepsilon}[M^{(k)}, M^{(j)}], \tag{29}$$

$$\frac{\partial \tilde{M}^{(j)}}{\partial \tilde{t}_k} - \frac{\partial \tilde{M}^{(k)}}{\partial \tilde{t}_j} = \frac{1}{\varepsilon}[\tilde{M}^{(j)}, \tilde{M}^{(k)}], \tag{30}$$

$$\frac{\partial M^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} = \frac{1}{\varepsilon}[M^{(j)}, \tilde{M}^{(k)}], \tag{31}$$

$k, j \in \mathbb{N}$, *for the equations* (26) *and* (27) *are fulfilled.*

Taking formally the small ε limit of the Toda lattice hierarchy, the shift operator $e^{\varepsilon\partial_{t_0}}$ is replaced by a variable u and the scaled commutator $\frac{1}{\varepsilon}[\cdot,\cdot]$ becomes a Poisson bracket with respect to the canonical variables $\log u$ and t_0. This leads us to the following definition of the *dispersionless Toda lattice hierarchy.*

Definition 6 ([37])**.** *The dispersionless Toda lattice hierarchy is given by the system of equations*

$$\frac{\partial z}{\partial t_k} = \{M_k, z\}, \qquad \frac{\partial \tilde{z}}{\partial t_k} = \{M_k, \tilde{z}\},$$

$$\frac{\partial z}{\partial \tilde{t}_k} = \{z, \tilde{M}_k\}, \qquad \frac{\partial \tilde{z}}{\partial \tilde{t}_k} = \{\tilde{z}, \tilde{M}_k\},$$

$k \in \mathbb{N}$, *for functions* z *and* $\tilde{z}$ *of* u, t_0, $\underline{t} = (t_k)_{k=1}^{\infty}$, *and* $\underline{\tilde{t}} = (\tilde{t}_k)_{k=1}^{\infty}$ *of the form*

$$z(u, t_0, \underline{t}, \underline{\tilde{t}}) = r(t_0, \underline{t}, \underline{\tilde{t}})u + \sum_{j=0}^{\infty} a_j(t_0, \underline{t}, \underline{\tilde{t}})u^{-j} \quad \textit{and}$$

$$\tilde{z}(u, t_0, \underline{t}, \underline{\tilde{t}}) = \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})u^{-1} + \sum_{j=0}^{\infty} \tilde{a}_j(t_0, \underline{t}, \underline{\tilde{t}})u^{j}.$$

Denoting with the subscripts $+$, 0, *and* $-$ *the positive, constant and negative part of a function considered as power series in* u,

$$M_k = (z^k)_+ + \frac{1}{2}(z^k)_0 \quad \textit{and} \quad \tilde{M}_k = (\tilde{z}^k)_- + \frac{1}{2}(\tilde{z}^k)_0, \quad k \in \mathbb{N}, \tag{32}$$

and the Poisson bracket is defined as

$$\{f, g\} = u\frac{\partial f}{\partial u}\frac{\partial g}{\partial t_0} - u\frac{\partial f}{\partial t_0}\frac{\partial g}{\partial u}. \tag{33}$$

The compatibility relations for the dispersionless Toda lattice hierarchy are also given as the limit $\varepsilon \to 0$, $n \to \infty$ of the compatibility relations (29) and (30).

Replacing the scaled commutators $\frac{1}{\varepsilon}[\cdot,\cdot]$ with the Poisson brackets $\{\cdot,\cdot\}$ in Proposition 1 we will be able to prove the following

Proposition 2 ([37])**.** *Let z and $\tilde{z}$ be a solution of the dispersionless Toda lattice hierarchy. Then M_k and $\tilde{M}_k$ given by* (32) *satisfy the following equations:*

$$\frac{\partial M_j}{\partial t_k} - \frac{\partial M_k}{\partial t_j} = \{M_k, M_j\}, \qquad \frac{\partial \tilde{M}_j}{\partial \tilde{t}_k} - \frac{\partial \tilde{M}_k}{\partial \tilde{t}_j} = \{\tilde{M}_j, \tilde{M}_k\},$$
$$\frac{\partial M_j}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}_k}{\partial t_j} = \{M_j, \tilde{M}_k\}, \qquad k, j \in \mathbb{N}.$$

Remark. *The two dimensional dispersionless Toda lattice equation* (25) *can be obtained from the compatibility equation*

$$\frac{\partial M_1}{\partial \tilde{t}_1} + \frac{\partial \tilde{M}_1}{\partial t_1} = \{M_1, \tilde{M}_1\}$$

to obtain

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right) = \frac{\partial^2}{\partial t_0^2}\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right) \;,$$

hence it is easy to recover (25) *by setting $a(t_0, t_1, \tilde{t}_1) = \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right)$.*

The same can be done for the Toda case. The two dimensional Toda lattice equation can be obtained from the compatibility condition (31) *with $j = k = 1$ and by setting $a(t_0, t_1, \tilde{t}_1) = \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right)$.*

Consider an $(n,m)-$polynomial curve parametrized by a conformal map of the form (23), uniquely characterized by the harmonic moments $(t_k)_{k=1}^{\infty}$ and the total mass t_0, and set

$$z(u, \underline{t}) = f(u) \qquad \tilde{z}(u, \underline{t}) = \overline{f}(u^{-1}) \;,$$

where $\underline{t} := (t_k)_{k=0}^{\infty} \cup (\bar{t}_k)_{k=0}^{\infty}$. Notice that here the role of $\tilde{t}_k$'s is assumed by the complex conjugate harmonic moments $(\bar{t}_k)_{k=0}^{\infty}$. We will show that z and $\tilde{z}$ satisfy dispersionless Toda as for the case presented in [11], but with a different string equation[2]:

Proposition 3. *We have the following* string equation

$$\{z^n, \tilde{z}^n\}(u, \underline{t}) = n \;, \tag{34}$$

where the Poisson bracket $\{\cdot,\cdot\}$ is the one defined in (33)*.*

Proposition 4. *We have for $1 \le k \le m$*

$$\frac{\partial z}{\partial t_k} = \{M_k, z\} \qquad \frac{\partial \tilde{z}}{\partial t_k} = \{M_k, \tilde{z}\} \qquad \frac{\partial z}{\partial \bar{t}_k} = \{z, \tilde{M}_k\} \qquad \frac{\partial \tilde{z}}{\partial \bar{t}_k} = \{\tilde{z}, \tilde{M}_k\} \;,$$

where

$$M_k(u, \underline{t}) := \left(z^k(u, \underline{t})\right)_+ + \frac{1}{2}\left(z^k(u, \underline{t})\right)_0 \qquad \left(\tilde{z}^k(u, \underline{t})\right)_- + \frac{1}{2}\left(\tilde{z}^k(u, \underline{t})\right)_0$$

[2]The string equation which appears in [11] coincide with (34) in the case $n = 1$.

Example. *Let us consider now the simple case with $P(z) = z^n$: the Schwarz function can be written explicitly and hence we can verify all the statement above directly.*

The Schwarz function and the functions z and $\tilde{z}$ assume the form

$$S^n(z) = nt_n + \frac{t_0}{z^n - n\bar{t}_n} = z^{-n+1}\left[nt_n z^{n-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty}(n\bar{t}_n)^k t_0 z^{-kn-1}\right] \ ,$$

and

$$z^n(u, t_0, t_n, \bar{t}_n)) = (t_0)^{\frac{1}{2}}u^n + n\bar{t}_n \qquad \tilde{z}^n(u, t_0, t_n, \bar{t}_n) = (t_0)^{\frac{1}{2}}u^{-n} + nt_n \ ,$$

where we used the notation introduced above.

With this explicit formulae we can verify directly Propositions 3 and 4. Indeed we have

$$\{z^n, \tilde{z}^n\}(u, t_0, t_n, \bar{t}_n) = u\frac{\partial z^n}{\partial u}\frac{\partial \tilde{z}^n}{\partial t_0} - u\frac{\partial z^n}{\partial t_0}\frac{\partial \tilde{z}^n}{\partial u} = nu^n(t_0)^{\frac{1}{2}}\frac{u^{-n}}{2(t_0)^{\frac{1}{2}}} + nu\frac{u^n}{2(t_0)^{\frac{1}{2}}}(t_0)^{\frac{1}{2}}u^{-n-1} = n \ .$$

In order to verify the equations (consider for simplicity the first two) of Proposition 4, notice that we just need to check their analogues with z^n and $\tilde{z}^n$ in place of z and $\tilde{z}$ respectively. Since in this simplified case we are considering only the times t_n and $\bar{t}_n$, this is the same as taking $M_k = 0$ for $k \neq n$, so we just need to compute

$$M_n = (t_0)^{\frac{1}{2}}u^n + \frac{1}{2}nt_n \ ,$$

in order to verify

$$\begin{aligned}
\{M_n, \tilde{z}^n\} &= \left\{z^n - \frac{1}{2}n\bar{t}_n, \tilde{z}^n\right\} = \{z^n, \tilde{z}^n\} = n = \frac{\partial \tilde{z}^n}{\partial t_n} \\
\{M_n, z^n\} &= \left\{z^n - \frac{1}{2}n\bar{t}_n, z^n\right\} = \{z^n, z^n\} = 0 = \frac{\partial z^n}{\partial t_n} \ .
\end{aligned}$$

Chapter 1

Equilibrium measure

In this chapter we will characterize the equilibrium measure for certain classes of potentials which appears as natural generalizations of the ones treated in [12, 11].

The contents of this chapter will follow [4].

1.1 Variational problem

Let us consider a normal matrix model with unitary invariant probability measures of the form

$$\mathcal{P}(M)dM = \frac{1}{\mathcal{Z}_N} e^{-\operatorname{tr} N\mathcal{V}(M)} dM \ . \tag{1.1}$$

The asymptotic analysis of various statistics of the eigenvalues in the large N limit leads to a logarithmic energy problem with external potential $\mathcal{V}(z)$ in the complex plane, which amounts to finding the minimizer of the *electrostatic energy functional*

$$\mathcal{I}_{\mathcal{V}}(\mu) = \iint \log \frac{1}{|z-w|} d\mu(z)d\mu(w) + \frac{1}{T}\int \mathcal{V}(z)d\mu(z)$$

in the space of probability measures in $\mathbb{C}$ (see [44, 12]). Following the terminology of [33], we say that $\mathcal{V}\colon \mathbb{C} \to (-\infty, \infty]$ is an *admissible potential* if it is lower semi-continuous, the set $\{z \colon \mathcal{V}(z) < \infty\}$ is of positive capacity and

$$\mathcal{V}(z) - \log|z| \to \infty \qquad \text{as } z \to \infty \ .$$

As shown in [33], for an admissible potential $\mathcal{V}$ there exists a unique measure $\mu = \mu_{\mathcal{V}}$ that minimizes the functional $\mathcal{I}_{\mathcal{V}}$, referred to as the *equilibrium measure* corresponding to $\mathcal{V}$. Moreover, $\mu_{\mathcal{V}}$ is the unique compactly supported measure of finite logarithmic energy for which the variational conditions

$$\frac{1}{T}\mathcal{V}(z) + 2\int \log\frac{1}{|z-w|} d\mu(w) = \ell_{2D} \quad z \in \operatorname{supp}(\mu) \quad \text{quasi-everywhere} \tag{1.2}$$

$$\frac{1}{T}\mathcal{V}(z) + 2\int \log\frac{1}{|z-w|} d\mu(w) \geq \ell_{2D} \quad z \in \mathbb{C} \quad \text{quasi-everywhere} \tag{1.3}$$

are valid for some constant ℓ_{2D} called *Robin constant* (a property is said to hold *quasi-everywhere* on a set S if it is valid at all points of S minus a set of zero logarithmic capacity).

By differentiating (1.2) it is easy to find the density of $\mu_{\mathcal{V}}$ with respect to the Lebesgue measure dA in the complex plane as

$$d\mu_{\mathcal{V}}(z) = \frac{1}{4\pi T}\Delta\mathcal{V}(z)\chi_D(z)dA(z) \ ,$$

where $D = \operatorname{supp}(\mu_{\mathcal{V}})$ and χ_D stands for the characteristic function of D.

The main problem is thus to characterize the support D of the equilibrium measure $\mu_{\mathcal{V}}$: thic can be done by expressing it in terms of its exterior uniformizing map

$$f\colon \{u \in \mathbb{C}\colon \ |u| > 1\} \to \mathbb{C} \setminus D$$

analytic and univalent for $|u| > 1$ and fixed by the asymptotic behaviour

$$f(\infty) = \infty \ , \quad f(u) = ru\left(1 + \mathcal{O}\left(\frac{1}{u}\right)\right) \quad u \to \infty \ , \tag{1.4}$$

where r is a real and positive parameter, called the *conformal radius* of D.

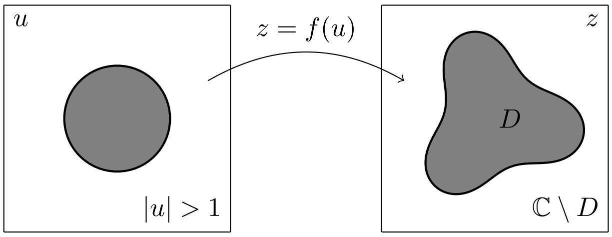


Figure 1.1: Illustration of the exterior conformal map

The equilibrium measure of the matrix model (1.1) is known explicitly only for some special cases, including circular symmetric potentials [33, 15], and perturbed quadratic potentials of the form

$$\mathcal{V}(z) = |z|^2 + h(z) + \overline{h(z)} \ ,$$

where $h(z)$ is either a polynomial (see [18, 19] for the unperturbed case, [10] for the quadratic case, [40, 12, 11, 6] for higher degree polynomials, where a cut-off needs to be imposed in order to ensure the convergence of the matrix integral) or the real part of a logarithm (see [2]).

1.2 Singularity correspondence

The equilibrium problem for the class of radially symmetric potential with polynomial perturbation was studied in [14] using conformal mapping techniques. Their method is based on a singularity correspondence result originating in the works of Richardson [32] and Gustafsson [20] and developed further by Entov and Etingof in [13]. Etingof and Ma obtained the functional form of the exterior conformal map for small perturbation.

We briefly recall the method developed by Entov and Etingof in [13] based on the singularity correspondence of Richardson [32] and Gustafsson [20] to find the functional form of the exterior uniformizing map of the support of the equilibrium measure.

To this end, consider potential of the form

$$\mathcal{V}(z) = W(z\bar{z}) - P(z) - \overline{P(z)}$$

where $W(x)$ is a real-valued twice continuously differentiable function on $(0, \infty)$ and $P(z)$ is a polynomial of degree m such that the potential $\mathcal{V}(z)$ is admissible. We seek the equilibrium measure in the form

$$d\mu(z) = \frac{1}{\pi T}\chi_D(z)\rho(z\bar{z})dA(z) \ ,$$

where

$$\rho(z\bar{z}) = \frac{1}{4}\Delta W(z\bar{z}) = W''(z\bar{z})z\bar{z} + W'(z\bar{z}) \ ,$$

and we assume that D is simply connected and ∂D is a Jordan curve with interior and exterior D_+ and D_- respectively. For any $z \in D_+$ we have

$$W'(z\bar{z})\bar{z} = \frac{1}{2\pi i}\int_{\partial D}\frac{W'(w\bar{w})\bar{w}dw}{w-z} + \frac{1}{2\pi i}\int_{D}\frac{\rho(w\bar{w})dw \wedge d\bar{w}}{w-z} \tag{1.5}$$

by Stokes' Theorem. Consider the Sokhotsky-Plemelj decomposition

$$W'(z\bar{z})\bar{z} = \varphi_+(z) - \varphi_-(z) \qquad z \in \partial D \ , \tag{1.6}$$

with

$$\varphi_\pm(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{W'(w\bar{w})\bar{w}dw}{w-z} \quad z \in D_\pm$$

holomorphic in $D_\pm$ respectively and $\varphi_-(z) \to 0$ for $z \to \infty$. From (1.5),

$$\frac{1}{\pi}\int_{\partial D}\frac{\rho(w\bar{w})dA(w)}{w-z} = \begin{cases} \varphi_+(z) - W'(z\bar{z})\bar{z} & z \in D_+ \\ \varphi_-(z) & z \in D_- \ . \end{cases} \tag{1.7}$$

Let us define the *logarithmic potential* of the measure μ

$$U^\mu(z) := \int \log\frac{1}{|z-w|}d\mu(w) \ .$$

By the variational equality (1.2) the effective potential

$$E(z) = \frac{1}{T}\mathcal{V}(z) + 2U^\mu(z),$$

is constant on D and therefore its ∂_z-derivative gives, by (1.7),

$$P'(z) = \varphi_+(z) \ . \tag{1.8}$$

This equation determines the shape of D through its external conformal map, as shown below.

In what follows, we assume that the uniformizing map $z = f(u)$ extends continuously to the boundary $|u| = 1$. The Sokhotsky-Plemelj decomposition (1.6) implies the identity

$$\varphi_+(f(u)) - \varphi_-(f(u)) = W'\left(f(u)\overline{f(u)}\right)\overline{f(u)} \qquad |u| = 1$$

on the unit circle of the u-plane. Given that

$$\overline{f(u)} = \bar{f}\left(\frac{1}{u}\right) \quad |u| = 1 \ ,$$

multiplying both sides by $f(u)$ and rearranging gives that

$$f(u)\varphi_+(f(u)) - H(u) = f(u)\varphi_-(f(u)) \qquad |u| = 1 \ , \tag{1.9}$$

where

$$H(u) = W'\left(f(u)\bar{f}\left(\frac{1}{u}\right)\right) f(u)\bar{f}\left(\frac{1}{u}\right) \ . \tag{1.10}$$

The main idea of the singularity correspondence is that since $f(u)\varphi_-(f(u))$ is holomorphic outside the unit circle $|u| = 1$, the l.h.s of (1.9), seen as a function on the unit circle, possesses an analytic continuation outside the unit disk of the u-plane. Therefore the analytic continuations of the functions $f(u)\varphi_+(f(u))$ and $H(u)$ have the same singularities in the exterior of the unit disk on the u-plane.

Since $f(u)$ is a univalent uniformizing map, the singularities of $z\varphi_+(z)$ in D_- are in one-to-one correspondence with the singularities of $f(u)\varphi_+(f(u))$ in $|u| > 1$ through the mapping f. This establishes a $1-1$ correspondence between the singularities of $z\varphi_+(z)$ in D_- and $H(u)$ in $|u| > 1$. In particular, $H(u)$ has a pole of order k at $u = u_0$ in $|u| > 1$ if and only if $z\varphi_+(z)$ has a pole of order k at $z_0 = f(u_0)$.

Since $z\varphi_+(z) = zP'(z)$ is a polynomial of degree m, it only has a pole of order m at $z = \infty$, and hence the only singularity of $H(u)$ in $|u| > 1$ is a pole of order m at $u = \infty$. Since

$$\overline{H(u)} = \bar{H}\left(\frac{1}{u}\right) = H(u) \qquad |u| = 1 \ ,$$

$H(u)$ is real on the unit circle and it can be analytically continued using the Schwarz reflection principle:

$$\tilde{H}(u) = \begin{cases} H(u) & |u| > 1 \\ \bar{H}\left(\frac{1}{u}\right) & |u| < 1 \ . \end{cases}$$

Therefore $\tilde{H}(u)$ is a Laurent polynomial of degree m which is real-valued on the unit circle. Since $\tilde{H}(u) \geq 0$ for the class of potentials considered, the Fejér-Riesz Lemma implies that there exists a polynomial $R(u)$ of degree m with zeros inside the unit disk such that

$$H(u) = R(u)\bar{R}\left(\frac{1}{u}\right) \ . \tag{1.11}$$

We seek the conformal map in the form

$$f(u) = rug(u) \ ,$$

where g is holomorphic in $|u| > 1$ with

$$g(u) = 1 + \mathcal{O}\left(\frac{1}{u}\right) \qquad u \to \infty \ ,$$

and

$$g(u) \neq 0 \qquad |u| > 1$$

since $z = 0$ is assumed to belong to D.

Consider now the case in which $W(x) = x^n$ and $m \leq 2n$: the singularity correspondence (1.11) implies that

$$n\left[f(u)\overline{f}\left(\frac{1}{u}\right)\right]^n = R(z)\overline{R}\left(\frac{1}{u}\right) \tag{1.12}$$

where R is a polynomial of degree m with zeros inside the unit disk, and therefore

$$nr^{2n}\left[g(u)\overline{g}\left(\frac{1}{u}\right)\right]^n = R(z)\overline{R}\left(\frac{1}{u}\right) \ .$$

Taking R of the form

$$R(u) = r^n\sqrt{n}\left(u^m + \sum_{j=0}^{m-1}\alpha_{m-j}u^j\right)$$

we get

$$(g(u))^n = 1 + \sum_{j=1}^{m}\frac{\alpha_j}{u^j} \ .$$

Hence we have

$$f(u) = ru\left(1 + \sum_{j=1}^{m}\frac{\alpha_j}{u^j}\right)^{\frac{1}{n}} \ . \tag{1.13}$$

The assumptions that the domain D is simply connected and that $z = 0$ belong to D are assured as far as the $|\alpha_j|$'s are small enough.

1.3 Potentials with discrete rotational symmetry

In the following part of the chapter we will focus on potentials of the form

$$\mathcal{V}(z) = |z|^{2n} - tz^d - \bar{t}\bar{z}^d \ , \tag{1.14}$$

where n and d are positive integers with $2n \geq d$, $T > 0$ and

$$\begin{cases} t \in \mathbb{C} & \text{if } d < 2n \\ |t| < 1/2 & \text{if } d = 2n \ . \end{cases}$$

For these potentials we are able to extend the construction of the conformal map of [14] reviewd above for all the admissible values of t: we show that there is a critical value $|t| = t_c$ such that the support of the equilibrium measure is simply connected for $|t| < t_c$

and has d connected components for $|t| > t_c$, and we determine explicitly the support of the equilibrium measure.

By a simple rotation of the variable z the analysis can be reduced to the case of real and positive t. Therefore, without loss of generality, we may and do assume that $t \in \mathbb{R}_+$, that is,

$$\mathcal{V}(z) = |z|^{2n} - tz^d + t\bar{z}^d \ .$$

The symmetry of these potentials allows us to simplify the problem of determining the support of the equilibrium measure by considering a simpler potential, whose equilibrium measure has simply connected support for all values of the parameters, and therefore it is given by a conformal mapping.

It is easy to see that the potentials in (1.14) are admissible. Moreover, these functions are invariant under the group of discrete rotations of order d:

$$\mathcal{V}\left(e^{\frac{2\pi i k}{d}} z\right) = \mathcal{V}(z) \qquad k = 0, \ldots, d-1 \ .$$

It is natural to expect that the corresponding equilibrium measure is invariant under the same group of symmetries, which motivates the following construction.

Definition 1.1. *For a fixed positive integer d and a Borel probability measure μ the associated d-*fold rotated measure *$\mu^{(d)}$ is defined to be*

$$\mu^{(d)} = \frac{1}{d}\sum_{k=0}^{d-1} \mu_k^{(d)} \ ,$$

where the kth summand is given by

$$\mu_k^{(d)}(B) = \mu\left(\varphi_k^{-1}(B \cap S_k)\right)$$

for any Borel set $B \subseteq \mathbb{C}$, where

$$S_k = \left\{ z \in \mathbb{C} \, : \, \frac{2\pi k}{d} \leq \arg(z) < \frac{2\pi(k+1)}{d} \right\} \qquad k = 0, \ldots, d-1$$

and

$$\varphi_k \colon \mathbb{C} \to S_k \ , \quad \varphi_k(re^{i\theta}) = r^{\frac{1}{d}} e^{\frac{i\theta}{d}} e^{\frac{2\pi i k}{d}} \ , \quad k = 0, \ldots, d-1 \ .$$

Remark. *Note that $z \in \operatorname{supp}(\mu^{(d)})$ iff $z^d \in \operatorname{supp}(\mu)$.*

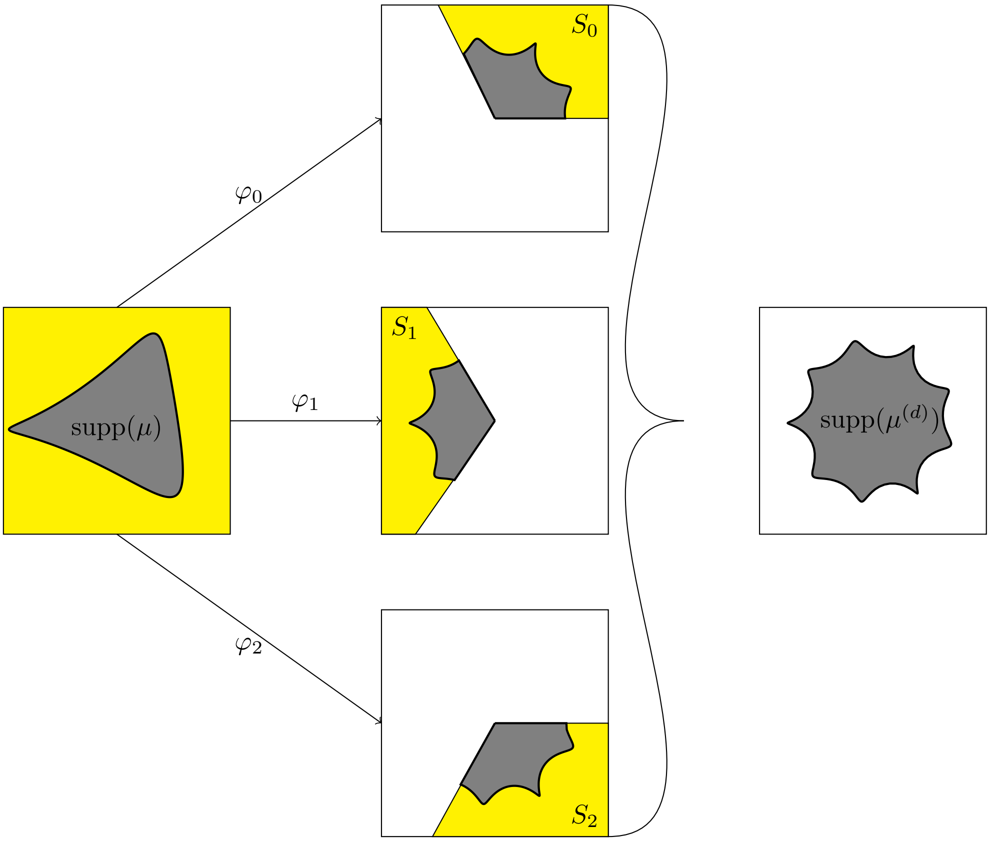


Figure 1.2: Illustration of the support of $\mu^{(d)}$ for $d = 3$

Lemma 1.1. *If the admissible potential $\mathcal{V}(z)$ can be written in terms of another admissible potential $Q(z)$ as*

$$\mathcal{V}(z) = \frac{1}{d} Q(z^d)$$

for some positive integer d, then the equilibrium measure for $\mathcal{V}$ is given by the d-fold rotated equilibrium measure of Q, i.e.,

$$\mu_{\mathcal{V}} = \mu_Q^{(d)} \ .$$

Proof. Let μ_Q be the equilibrium measure of the admissible potential $Q(z)$, which is uniquely characterized by the following variational inequalities for some constant ℓ_{2D}:

$$\begin{aligned} Q(z) + 2U^{\mu_Q}(z) &= \ell_{2D} \quad z \in \mathrm{supp}(\mu_Q) \quad \text{q.e.} \\ Q(z) + 2U^{\mu_Q}(z) &\geq \ell_{2D} \quad z \notin \mathrm{supp}(\mu_Q) \quad \text{q.e.} \end{aligned}$$

Therefore

$$\begin{aligned} Q(z^d) + 2U^{\mu_Q}(z^d) &= \ell_{2D} \quad z^d \in \mathrm{supp}(\mu_Q) \quad \text{q.e.} \\ Q(z^d) + 2U^{\mu_Q}(z^d) &\geq \ell_{2D} \quad z^d \notin \mathrm{supp}(\mu_Q) \quad \text{q.e.} \end{aligned}$$

Using the previous proposition with $\mu = \mu_Q^{(d)}$, we find that

$$\begin{aligned} \mathcal{V}(z) + 2U^{\mu_Q^{(d)}}(z) &= \frac{\ell_{2D}}{d} \quad z \in \operatorname{supp}(\mu_Q^{(d)}) \quad \text{q.e.} \\ \mathcal{V}(z) + 2U^{\mu_Q^{(d)}}(z) &\geq \frac{\ell_{2D}}{d} \quad z \notin \operatorname{supp}(\mu_Q^{(d)}) \quad \text{q.e.} \quad , \end{aligned}$$

where

$$Q(z) = \frac{1}{d}\mathcal{V}(z^d) \ .$$

This is enough to prove that $\mu_{\mathcal{V}} = \mu_Q^{(d)}$. □

As a corollary, the problem of finding the equilibrium measure for the class (1.14) reduces to the study of the following family of potentials:

$$Q(z) = d\left(|z|^{\frac{2n}{d}} - tz - t\bar{z}\right) \ . \tag{1.15}$$

For this choice of the potential the formula 1.1 for the equilibrium measure becomes

$$d\mu_Q(z) = \frac{1}{4\pi T}\Delta Q(z)\chi_K(z)dA(z) = \frac{n^2}{\pi T d}|z|^{\frac{2n}{d}-2}\chi_K(z)dA(z) \ ,$$

where $K = \operatorname{supp}(\mu_Q)$ and χ_K stands for the characteristic function of K.

Remark. *If the support K of the measure μ contains $z = 0$ and it is given by exterior uniformizing map*

$$f(u) = rug(u)$$

with

$$g(u) \neq 0 \quad |u| > 1 \ , \qquad g(u) = 1 + \mathcal{O}\left(\frac{1}{u}\right) \quad u \to \infty$$

then the support of $\mu^{(d)}$ has the conformal map

$$u \mapsto f(u^d)^{\frac{1}{d}} = r^{\frac{1}{d}}ug(u^d)^{\frac{1}{d}}$$

which is well-defined, univalent in $|u| > 1$ and normalized according to (1.4).

Thus the equilibrium measures of $\mathcal{V}$ and Q are easily related in a very simple way. The *logarithmic potential* and the *Cauchy transform* of a measure μ, defined as

$$U^\mu(z) = \int \log\frac{1}{|z-w|}d\mu(w) \ ,$$

and

$$C_\mu(z) = \int \frac{d\mu(w)}{w-z} \ ,$$

respectively, can also be easily computed from their symmetry reduced version.

Proposition 1.1. *The d-fold rotated measure $\mu^{(d)}$ of the measure μ has the logarithmic potential*

$$U^{\mu^{(d)}}(z) = \frac{1}{d}U^\mu(z^d)$$

and the Cauchy transform

$$C_{\mu^{(d)}}(z) = z^{d-1}C_\mu(z^d) \ .$$

Proof.

$$U^{\mu^{(d)}}(z) = \frac{1}{d}\sum_{k=0}^{d-1}\int_{S_k} \log\frac{1}{|z-w|}d\mu_k^{(d)}(w)$$
$$= \frac{1}{d}\sum_{k=0}^{d-1}\int_{\mathbb{C}} \log\frac{1}{|z-\varphi_k(u)|}d\mu(u) \ .$$

Since

$$\prod_{k=0}^{d-1}(z-\varphi_k(u)) = z^d - u \ ,$$

We have that

$$U^{\mu^{(d)}}(z) = \frac{1}{d}\int_{\mathbb{C}} \log\frac{1}{|z^d-u|}d\mu(u) = \frac{1}{d}U^{\mu}(z^d) \ .$$

The equation involving the Cauchy transforms can be proved similarly. □

1.4 The conformal map

Now we construct the conformal map for potentials of the form (1.15) by applying the procedure described in Section 1.2 for the specific choice:

$$W(x) = d\,x^{\frac{n}{d}} \quad \text{and} \quad P(z) = td\,z \ .$$

It is important to distinguish between two different regimes corresponding to small values of t and large values of t. If t is small enough we expect that $z=0$ belongs to $K = \mathrm{supp}(\mu_Q)$ since the equilibrium measure for $t=0$ is supported on a disk centered at the origin. For large t, however, the constant field coming from the perturbative term $tz + t\bar{z}$ becomes strong enough so that the equilibrium domain separates from the origin. We expect a critical transition when $0 \in \partial K$ that will affect the functional form of the conformal map. Hence the cases $0 \in K$ and $0 \in \mathbb{C}\setminus K$ are referred to as *pre-critical* and *post-critical*, respectively.

1.4.1 Pre-critical case

Following [14], we seek the conformal map g in the form

$$f(u) = rug(u) \ ,$$

where g is holomorphic in $|u|>1$ with

$$g(u) = 1 + \mathcal{O}\left(\frac{1}{u}\right) \quad u\to\infty \ ,$$

and

$$g(u) \neq 0 \qquad |u|>1$$

since $z=0$ is assumed to belong to K. The singularity correspondence (1.11) gives equation 1.12:

$$n\left[f(u)\bar{f}\left(\frac{1}{u}\right)\right]^{\frac{n}{d}} = R(u)\bar{R}\left(\frac{1}{u}\right)$$

for some linear polynomial $R(u)$ and therefore

$$nr^{\frac{2n}{d}}\left[g(u)\bar{g}\left(\frac{1}{u}\right)\right]^{\frac{n}{d}} = R(u)\bar{R}\left(\frac{1}{u}\right) \ .$$

The polynomial R is parametrized as

$$R(u) = r^{\frac{n}{d}}\sqrt{n}(u-\alpha) \ ,$$

where $|\alpha| \leq 1$ is an unknown parameter, to be determined from the data. Therefore g is of the form

$$g(u) = \left(1-\frac{\alpha}{u}\right)^{\frac{d}{n}} \ ,$$

and hence

$$f(u) = ru\left(1-\frac{\alpha}{u}\right)^{\frac{d}{n}} \ . \tag{1.16}$$

To find the equlibrium support we have to determine the correct values of the parameters r and α. The analysis of the equations which give these parameters allows to find the critical value $t = t_c$ for which the condition $|\alpha| < 1$ is no longer satisfied; it will be shown that the value $t = t_c$ separates the pre- and post-critical cases.

Total mass condition.

$$\begin{aligned}
\frac{1}{\pi T}\int_K \rho(w\bar{w})dA(w) &= \frac{n}{T}\frac{1}{2\pi i}\int_{\partial K} w^{\frac{n}{d}}\bar{w}^{\frac{n}{d}}\frac{dw}{w} \\
&= \frac{n}{T}\frac{1}{2\pi i}\int_{|u|=1}\left[f(u)\bar{f}\left(\frac{1}{u}\right)\right]^{\frac{n}{d}}\frac{f'(u)}{f(u)}du \\
&= \frac{r^{\frac{2n}{d}}n}{T}\frac{1}{2\pi i}\int_{|u|=1}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\left[\left(1-\frac{d}{n}\right)\frac{1}{u}+\frac{d}{n}\frac{1}{u-\alpha}\right]du \\
&= \frac{r^{\frac{2n}{d}}}{T}\left(n+(n-d)|\alpha|^2\right) \ .
\end{aligned}$$

In the calculation above we assumed that $|\alpha| < 1$. In the chosen normalization of the energy problem the equilibrium measure μ_Q is a probability measure, and this gives the following equation in r and the modulus of α:

$$\frac{r^{\frac{2n}{d}}}{T}\left(n+(n-d)|\alpha|^2\right) = 1 \ . \tag{1.17}$$

Deformation condition. From (1.10) we can compute

$$\frac{H(u)}{f(u)} = nr^{\frac{2n}{d}}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{1}{f(u)} = -nr^{\frac{2n}{d}-1}\bar{\alpha}+\mathcal{O}\left(\frac{1}{u}\right) \quad u\to\infty \ .$$

Since we know φ_+ from (1.8), by using (1.9) and the previous asympotic behaviour we can obtain

$$\begin{aligned}
\varphi_+(f(u)) &= -nr^{\frac{2n}{d}-1}\bar{\alpha} \\
\varphi_-(f(u)) &= -nr^{\frac{2n}{d}-1}\bar{\alpha} - nr^{\frac{2n}{d}}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{1}{f(u)} \ .
\end{aligned} \tag{1.18}$$

Comparison of (1.18) with (1.8) gives the second equation

$$t = -\frac{n}{d} r^{\frac{2n}{d}-1} \alpha \ , \tag{1.19}$$

where we replaced $\bar{\alpha}$ with α, since we can easily deduce that α is real. Combining (1.17) and (1.19) gives the equation

$$r^{\frac{4n}{d}-2} - \frac{T}{n} r^{\frac{2n}{d}-2} + \frac{n-d}{n}\frac{d^2}{n^2} t^2 = 0 \tag{1.20}$$

for the conformal radius r as a function of t. It can be shown (see Appendix A.2) that (1.20) has a unique positive solution $r = r_0$ such that

$$|\alpha(r_0)| < 1 \ .$$

Equations (1.20) and (1.19) are what we need to characterize all the parameters in the pre-critical case.

We can now determine the critical value of t: from (1.20) and (1.19) we get

$$\alpha^2 = \frac{n}{n-d}\left(\frac{T}{n} r_0^{-\frac{2n}{d}} - 1\right) \ .$$

Solving the critical equation $\alpha = 1$ and we can obtain the critical values t_c and $r_c := r(t_c)$ given by

$$t_c = \frac{n}{d}\left(\frac{T}{2n-d}\right)^{\frac{2n-d}{2n}} \qquad r_c = \left(\frac{T}{2n-d}\right)^{\frac{d}{2n}} \ .$$

1.4.2 The special cases $d = n$ and $d = 2n$

The solution for the special cases $d = n$ and $d = 2n$ are well-known, but analyzing them in detail will be helpful in constructing the post-critical conformal map.

Case $d = n$

For $d = n$ the reduced potential is

$$Q(z) = n\left(|z|^2 - tz - t\bar{z}\right) = n|z-t|^2 - nt^2 \ ,$$

and therefore the equilibrium measure is the normalized area measure on the disk

$$K = \left\{ z : \ |z - t| \leq \sqrt{\frac{T}{n}} \right\} \ ,$$

with exterior conformal map

$$f(u) = ru + t = ru\left(1 + \frac{t}{r}\frac{1}{u}\right) \ , \qquad r = \sqrt{\frac{T}{n}} \ . \tag{1.21}$$

It is important to notice that this conformal map is well defined for any value of t, and hence the form we want for the post-critical conformal map should reduce to 1.21 for the $d = n$.

Case $d = 2n$

For $d = 2n$ we have

$$Q(z) = 2n\left(|z| - tz - t\bar{z}\right) \ ,$$

which is just a re-parametrization of the well-known case

$$\tilde{Q}(z) = \frac{1}{2}Q(z^2) = n\left(|z|^2 - tz^2 - t\bar{z}^2\right) \ .$$

The equilibrium measure for $\tilde{Q}$ is the normalized area measure on an ellipse with exterior conformal map [10, 40, 12]

$$\tilde{f}(u) = \sqrt{r}u\left(1 + \frac{2t}{u^2}\right) \ , \qquad r = \frac{T}{n}\frac{1}{1-4t^2} \ .$$

By applying Lemma 1.1 for $\tilde{Q} \to Q$ we get that the uniformizing map for the support of μ_Q is

$$f(u) = \tilde{f}(\sqrt{u})^2 = ru\left(1 + \frac{2t}{u}\right)^2 \ .$$

Given the admissibility condition $t < \frac{1}{2}$ it is easy to see that $f(u)$ is univalent in $|u| > 1$.

1.4.3 Post-critical case

If the $z = 0$ is outside K then $f(u)$ must have a zero in $|u| > 1$. To make sense of (1.12), the ansatz presented in [14] needs to be modified to account for the vanishing of the conformal map at, say, $u = \beta$. This can be done by introducing a *Blaschke factor*, i.e.,

$$u \mapsto \frac{u - \beta}{1 - \bar{\beta}u} \qquad |\beta| > 1 \ ,$$

that leaves the unit circle of the u-plane invariant and swaps the interior and the exterior. This function can be factored into the proposed form of the conformal map:

$$f(u) = ru\left(-\bar{\beta}\right)\frac{u - \beta}{1 - \bar{\beta}u}g(u)$$

where g is holomorphic in $|u| > 1$ with

$$g(u) \neq 0 \qquad |u| > 1 \ , \qquad g(u) = 1 + \mathcal{O}\left(\frac{1}{u}\right) \qquad u \to \infty \ .$$

Then

$$f(u)\bar{f}\left(\frac{1}{u}\right) = r^2|\beta|^2\frac{u-\beta}{1-\bar{\beta}u}g(u)\frac{1-\bar{\beta}u}{u-\beta}\bar{g}\left(\frac{1}{u}\right) = r^2|\beta|^2 g(u)\bar{g}\left(\frac{1}{u}\right)$$

and therefore

$$nr^{\frac{2n}{d}}|\beta|^{\frac{2n}{d}}\left[g(u)\bar{g}\left(\frac{1}{u}\right)\right]^{\frac{n}{d}} = R(u)\bar{R}\left(\frac{1}{u}\right) .$$

Now

$$R(u) = r\beta n^{\frac{d}{2n}}(u-\alpha) ,$$

where $|\alpha| < 1$ is unknown. The function g is again

$$g(u) = \left(1 - \frac{\alpha}{u}\right)^{\frac{d}{n}} ,$$

and hence

$$f(u) = ru\frac{u-\beta}{u-\frac{1}{\bar{\beta}}}\left(1-\frac{\alpha}{u}\right)^{\frac{d}{n}} , \tag{1.22}$$

where the parameters r, α and β are to be determined from the data.

As for the pre-critical case, there are two equations but now the three unknowns are three: r, α and β; hence the conditions we have are insufficient to characterize the conformal map. However, the uniqueness of the equilibrium measure implies that there is only one choice of the parameters which corresponds to this unique measure. As stated above this selection problem will be solved by using the known $d = n$ case to make an ansatz on the value of β.

Total mass condition. Now the total mass of the measure is given by

$$\begin{aligned}\frac{1}{\pi T}\int_K \rho(w\bar{w})dA(w) &= \frac{|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}}n}{T}\frac{1}{2\pi i}\int_{|u|=1}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{f'(u)}{f(u)}du \\ &= \frac{n}{T}|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}}\left(\frac{\bar{\alpha}}{\bar{\beta}}+\frac{\alpha}{\beta}-\frac{d}{n}|\alpha|^2\right) .\end{aligned}$$

This gives the following equation on r and the modulus of α:

$$\frac{n}{T}|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}}\left(\frac{\bar{\alpha}}{\bar{\beta}}+\frac{\alpha}{\beta}-\frac{d}{n}|\alpha|^2\right) = 1 . \tag{1.23}$$

Deformation condition. Now

$$\frac{H(u)}{f(u)} = n|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{1}{f(u)} = -n|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}-1}\bar{\alpha}+\mathcal{O}\left(\frac{1}{u}\right) \quad u\to\infty ,$$

and therefore, applying the same comparison as in the pre-critical case we get

$$\begin{aligned}\varphi_+(f(u)) &= -n|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}-1}\bar{\alpha} \\ \varphi_-(f(u)) &= -n|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}-1}\bar{\alpha} - n|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{1}{f(u)} .\end{aligned}$$

This implies, by (1.8), that

$$t = -\frac{n}{d}|\beta|^{\frac{2n}{d}}r^{\frac{2n}{d}-1}\bar{\alpha} . \tag{1.24}$$

Equations (1.23) and (1.24) are not enough to determine all the parameters. To bypass this difficulty we make an ansatz on the parameter β and later on we will justify that the choice made is correct.

Ansatz on β and the parameters of the conformal map

To get an idea what the value of β should be, it is natural, at first, to look at the case $d = n$. As we have seen above, the corresponding conformal map is linear for all values of t:

$$f(u) = r\left(u + \frac{t}{r}\right) , \qquad r = \sqrt{\frac{T}{n}} .$$

The post-critical form (1.22) of the conformal map gives

$$f(u) = r\frac{u-\beta}{u-\frac{1}{\beta}}(u-\alpha) ,$$

which is linear if and only if $\beta = \frac{1}{\bar{\alpha}}$. This suggests to make the following ansatz:

$$\beta = \frac{1}{\bar{\alpha}} ,$$

which reduce the post-critical conformal map to the form

$$r\left(u - \frac{1}{\bar{\alpha}}\right)\left(1 - \frac{\alpha}{u}\right)^{\frac{d}{n}-1} .$$

With this assumption, the total mass and deformation conditions assume the following form:

$$\frac{n}{T} r^{\frac{2n}{d}} \frac{2n-d}{n} \alpha^{2-\frac{2n}{d}} = 1 \tag{1.25}$$

$$-\frac{n}{d} r^{\frac{2n}{d}-1} \alpha^{1-\frac{2n}{d}} = t , \tag{1.26}$$

Proposition 1.2. *where we simplified the formulae since it can be* a posteriori *deduced that α is real. Assuming $t > t_c$, the following explicit formulae hold:*

$$r = \left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{d}{n}t\right)^{\frac{d-n}{2n-d}} \qquad \alpha = -\left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{d}{n}t\right)^{-\frac{n}{2n-d}} .$$

Moreover, the condition $|\alpha| < 1$ is satisfied for all $t > t_c$.

Proof. From Eq. (1.26) we obtain

$$\alpha = \left(\frac{d}{n}t\right)^{-\frac{d}{2n-d}} r ,$$

which, with equation (1.25) gives the explicit form for r and α above. The condition

$$\alpha = \left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{d}{n}t\right)^{-\frac{n}{2n-d}} < 1$$

rewritten in terms of t is equivalent to

$$t > \frac{n}{d}\left(\frac{T}{2n-d}\right)^{\frac{2n-d}{2n}} = t_c ,$$

i.e., $|\alpha| < 1$ is always satisfied in the post-critical regime $t > t_c$. $\square$

Remark. *It is to be checked that these functions are univalent maps outside the unit circle: for this we refer to Appendix A.1.*

A straightforward application of Lemma 1.1 to the results obtained for the potential $Q(z)$ yields to analogous results for the potential $\mathcal{V}$, which can be summarized in the following

Theorem 1.1. *Let $0 < d < 2n$. The equilibrium measure for the potential*

$$\mathcal{V}(z) = |z|^{2n} - tz^d - t\bar{z}^d$$

is given by $\mu_Q^{(d)}$, the d-fold rotated equilibrium measure of Q given above.

More precisely, $\mu_{\mathcal{V}}$ is absolutely continuous with respect to the area measure with density

$$d\mu_{\mathcal{V}}(z) = \frac{n^2}{\pi T}|z|^{2n-2}\chi_D(z)dA(z) \ ,$$

and the support set D of $\mu_{\mathcal{V}}$ can be described as follows:

- *For $t \le t_c$, D is simply connected and it is given by the exterior conformal map*

$$g(u) = \left(f(u^d)\right)^{\frac{1}{d}} = r^{\frac{1}{d}} u \left(1 - \frac{\alpha}{u^d}\right)^{\frac{1}{n}} \ ,$$

  where $r = r(t)$ is a particular solution of the equation

$$r^{\frac{4n}{d}-2} - \frac{T}{n} r^{\frac{2n}{d}-2} + \frac{n-d}{n}\frac{d^2}{n^2}t^2 = 0 \ ,$$

  and

$$\alpha = -\frac{d}{n} t\, r^{-\frac{2n-d}{d}} \ .$$

- *For $t > t_c$, D has d disjoint simply connected components and their boundary is parametrized by*

$$g(u) = r^{\frac{1}{d}} u \left(1 - \frac{1}{\bar{\alpha}u^d}\right)^{\frac{1}{d}} \left(1 - \frac{\alpha}{u^d}\right)^{\frac{1}{n}-\frac{1}{d}} \ ,$$

  where u lies on the unit circle and the factor

$$\left(1 - \frac{1}{\bar{\alpha}u^d}\right)^{\frac{1}{d}} \sim 1 + \mathcal{O}\left(\frac{1}{u}\right) \quad u \to \infty$$

  is defined with some suitably defined branch cuts on $|u| \ge 1$, with r and α given by

$$r = \left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{d}{n}t\right)^{\frac{d-n}{2n-d}} \qquad \alpha = -\left(\frac{T}{2n-d}\right)^{\frac{1}{2}} \left(\frac{dt}{n}\right)^{-\frac{n}{2n-d}} \ .$$

Remark. *It is easy to check that the parameters $r = r(t)$ and $\alpha = \alpha(t)$ are continuous in*

t. Hence the conformal map, as a function of t, is also continuous:

$$\lim_{t\to t_c+} f(u) = r_c\left(u - \frac{1}{\overline{\alpha_c}}\right)\left(1 - \frac{\alpha_c}{u}\right)^{\frac{d}{n}-1} = r_c u\left(1 - \frac{\alpha_c}{u}\right)^{\frac{d}{n}} = \lim_{t\to t_c-} f(u)\ .$$

For $t < t_c$ the point $z = 0$ belongs to the support of the equilibrium measure and at $t = t_c$ the origin is on the boundary of the support.

However, it is reasonable to expect a "phase transition" near the critical time t_c for random matrix observables in a suitable scaling limit (see [15] in this direction for the case of circular symmetric potentials).

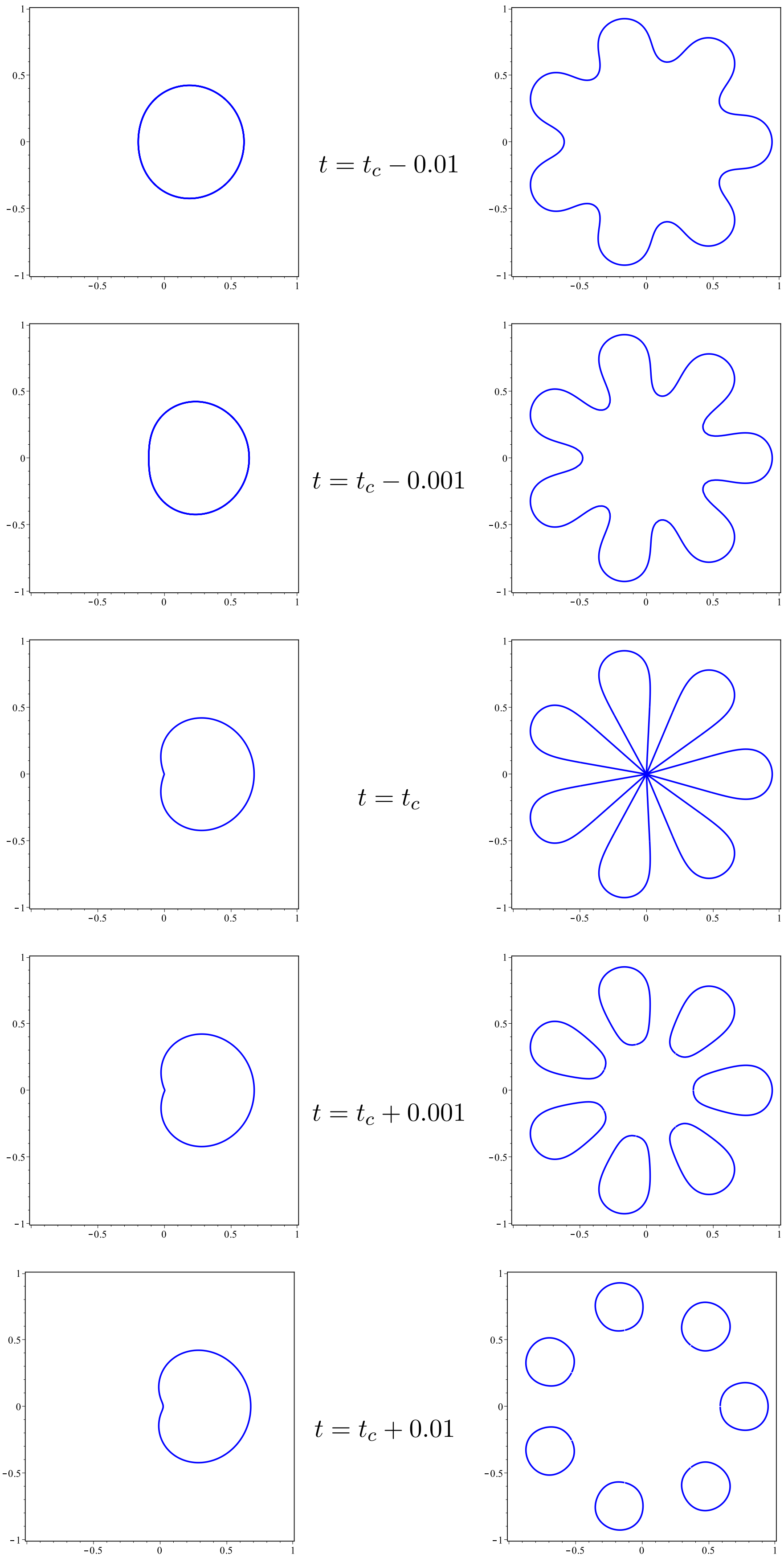


Figure 1.3: The equilibrium supports for $n = 9$, $d = 7$, $T = 1$ ($Q(z)$ left, $\mathcal{V}(z)$ right)

1.5 Variational inequalities

According to Proposition A.1, for $0 < t < t_c$ a conformal map of the form (1.16) satisfies the assertions of the theorem. The complementary result Prop. 1.2 shows that for $t > t_c$ the ansatz (1.22) with $\beta = \frac{1}{\bar{\alpha}}$ leads to a system of equations for the parameters that have a solution with the stated properties.

To conlude the proof of the results presented above we need to show that the variational inequality (1.3) holds in both cases for the proposed measures.

We have

$$T\,\partial_z E(z) = W'(z\bar{z})\bar{z} - P'(z) + \varphi_-(z) \qquad z \in D_- \ .$$

This is identically zero on ∂K. Since $E(z) = \ell_{2D}$ on the boundary of K and

$$E(z) \to \infty \qquad |z| \to \infty \ ,$$

the effective potential can go below the value F in D_- only if $E(z)$ has a critical point $z_0 \in D_-$ such that

$$T\,\partial_z E(z)|_{z=z_0} = W'(z_0\overline{z_0})\overline{z_0} - P'(z_0) + \varphi_-(z_0) = 0 \ . \tag{1.27}$$

Therefore to prove the variational inequality (1.3) it is sufficient to show that the derivative does not have any critical points in D_-.

To this end, it is enough to show that (1.27) is never satisfied, i.e., in terms of the uniformizing coordinate u,

$$T\,\partial_z E(z)|_{z=f(u)} = W'(f(u)\overline{f(u)})\overline{f(u)} - P'(f(u)) + \varphi_-(f(u)) \neq 0 \qquad |u| > 1 \ .$$

1.5.1 Pre-critical case

$$\begin{aligned} T\,\partial_z E(z)|_{z=f(u)} &= W'(f(u)\overline{f(u)})\overline{f(u)} - nr^{\frac{2n}{d}}\left(1 - \frac{\alpha}{u}\right)(1 - \bar{\alpha}u)\frac{1}{f(u)} \\ &= nr^{\frac{2n}{d}}\left(1 - \frac{\alpha}{u}\right)\frac{1}{f(u)}\left[|u|^{\frac{2n}{d}}\left(1 - \frac{\bar{\alpha}}{\bar{u}}\right) - (1 - \bar{\alpha}u)\right] \ . \end{aligned}$$

Since $|\alpha| < 1$, this expression vanishes for some u with $|u| > 1$ only if

$$|u|^{\frac{2n}{d}}\left(1 - \frac{\bar{\alpha}}{\bar{u}}\right) - (1 - \bar{\alpha}u) = 0 \ ,$$

which implies that

$$|u|^{\frac{2n}{d}-1} = \left|\frac{1 - \bar{\alpha}u}{u - \alpha}\right| \ .$$

This is impossible since

$$|u|^{\frac{2n}{d}-1} > 1 \quad \text{and} \quad \left|\frac{1 - \bar{\alpha}u}{u - \alpha}\right| < 1 \ .$$

1.5.2 Post-critical case

Here the derivative of the effective potential is

$$
\begin{aligned}
T\,\partial_z E(z)|_{z=f(u)} &= W'(f(u)\overline{f(u)})\overline{f(u)} - \frac{nr^{\frac{2n}{d}}}{|\alpha|^{\frac{2n}{d}}}\left(1-\frac{\alpha}{u}\right)(1-\bar{\alpha}u)\frac{1}{f(u)} \\
&= \frac{nr^{\frac{2n}{d}}}{|\alpha|^{\frac{2n}{d}}}\left(1-\frac{\alpha}{u}\right)\frac{1}{f(u)}\left[|u|^{\frac{2n}{d}}\left|\frac{1-\bar{\alpha}u}{u-\alpha}\right|^{\frac{2n}{d}}\left(1-\frac{\bar{\alpha}}{\bar{u}}\right)-(1-\bar{\alpha}u)\right] \;.
\end{aligned}
$$

Since $|\alpha| < 1$, this may vanish for some u in the exterior of the unit disk only if

$$
|u|^{\frac{2n}{d}}\left|\frac{1-\bar{\alpha}u}{u-\alpha}\right|^{\frac{2n}{d}}\left(1-\frac{\bar{\alpha}}{\bar{u}}\right)-(1-\bar{\alpha}u)=0\;,
$$

which implies

$$
\left|u\frac{1-\bar{\alpha}u}{u-\alpha}\right| = 1\;.
$$

As a consequence we have

$$
1-\bar{\alpha}u = 1-\frac{\bar{\alpha}}{\bar{u}}\;,
$$

and thus $|u| = 1$, a contradiction. The variational inequality is hence satisfied in both cases and this concludes the proof of the results above.

Chapter 2

Orthogonal polynomials

In this chapter we study, following [3], the strong asymptotic of the polynomials $p_n(\lambda)$[1] orthogonal with respect to the external potential of the form

$$e^{-N\mathcal{V}(\lambda)}, \quad \mathcal{V}(\lambda) = |\lambda|^{2s} - t\lambda^s - \bar{t}\bar{\lambda}^s \qquad \lambda \in \mathbb{C},\ s \in \mathbb{N}, \quad t \in \mathbb{C}^*,$$

where the potential $\mathcal{V}(\lambda)$ has an s-discrete rotational symmetry. Given the discrete rotational symmetry of the problem, we assume t real and positive without loss of generality, namely the potential $\mathcal{V}(\lambda)$ takes the form

$$e^{-N\mathcal{V}(\lambda)}, \quad \mathcal{V}(\lambda) = |\lambda|^{2s} - t(\lambda^s + \bar{\lambda}^s) \qquad \lambda \in \mathbb{C},\ s \in \mathbb{N}, \quad t > 0. \tag{2.1}$$

We briefly recall the results obtained in 1, which says that if the potential $\mathcal{V}(\lambda)$ can be written in the form

$$\mathcal{V}(\lambda) = \frac{1}{s}Q(\lambda^s)$$

then the equilibrium measure for $\mathcal{V}$ can be obtained from the equilibrium measure of Q by an unfolding procedure. In our particular case

$$Q(u) = s|u|^2 - st(u + \bar{u}) = s|u - t|^2 - st^2$$

corresponds to the Ginibre ensemble so that the equilibrium measure for the potential Q is the normalized area measure of the disk

$$|u - t| = t_c, \quad t_c = \sqrt{\frac{T}{s}},$$

where T has been defined in (2). The equilibrium measure for $\mathcal{V}$ turns out the be equal to

$$d\mu_{\mathcal{V}} = \frac{s}{\pi t_c^2}|\lambda|^{2(s-1)}\chi_D dA(\lambda) \tag{2.2}$$

where dA is the area measure and χ_D is the characteristic function of the domain D given by the equation

$$D := \{\lambda \in \mathbb{C},\ |\lambda^s - t| \le t_c\}. \tag{2.3}$$

[1]Notice that in this chapter we are using the letter λ to denote the variable in the unfolded plane in order to avoid confusion and to keep clear the changes of variables.

We observe that for $|t| < t_c$ the equation (2.3) describes a simply connected domain in the complex plane with uniformizing map from the exterior of the unit circle to the exterior of D given by

$$f(\zeta) = t_c^{\frac{1}{s}} \zeta \left(1 + \frac{t}{t_c}\frac{1}{\zeta^s}\right)^{\frac{1}{s}}$$

with inverse

$$F(\lambda) = \frac{\lambda}{t_c^{\frac{1}{s}}}\left(1 - \frac{t}{\lambda^s}\right)^{\frac{1}{s}}.$$

For $|t| > t_c$ the domain defined by the equation (2.3) is multiply connected and consists of s components which have a discrete rotational symmetry. For $s = 2$ the domain D is called Cassini oval. We observe that the boundary of the domain D can be also expressed by the equation

$$\bar{\lambda} = S(\lambda), \quad S(\lambda) = \left(t + \frac{t_c^2}{\lambda^s - t}\right)^{\frac{1}{s}} \tag{2.4}$$

The function $S(\lambda)$ is analytic in a neighbourhood of ∂D and it is called the *Schwarz function* associated to ∂D (see e.g. [8]).

Remark. *The domain D is a quadrature domain with respect to the measure $d\mu_{\mathcal{V}}$. Indeed for any analytic function $h(\lambda)$ in D one has, applying Stokes theorem and the residue theorem,*

$$\int_D h(\lambda) d\mu_{\mathcal{V}}(\lambda) = \frac{s}{4\pi T}\int_{\partial D} h(\lambda)\lambda^{s-1}(S(\lambda))^s d\lambda = \sum_{k=1}^{s} c_k h(\lambda_k)$$

where c_k are some fixed constants independent from the function $h(\lambda)$ and $\lambda_k = t^{\frac{1}{s}} e^{\frac{2\pi i(k-1)}{s}}$.

2.1 Symmetry reduction

In analogy with what we did in Chapter 1 we are going to simplify the problem using the symmetry of the potential. Indeed the $\mathbb{Z}_s$-symmetry of the orthogonality measure (2.1) is inherited by the corresponding orthogonal polynomials. Indeed the non trivial orthogonality relations are

$$\int_{\mathbb{C}} p_n(\lambda)\bar{\lambda}^{js+l} e^{-N\mathcal{V}(\lambda,\bar{\lambda})} dA(\lambda), \quad j = 0, \dots, k-1,$$

where k and l are such that

$$n = ks + l, \qquad 0 \leq l \leq s-1,$$

i.e., the n-th monic orthogonal polynomial satisfies the relation

$$p_n(e^{\frac{2\pi i}{s}}\lambda) = e^{\frac{2\pi i n}{s}} p_n(\lambda).$$

It follows that there exists a monic polynomial $q_k^{(l)}$ of degree k such that

$$p_n(\lambda) = \lambda^l q_k^{(l)}(\lambda^s). \tag{2.5}$$

Therefore the sequence of orthogonal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ can be split into d subsequences labelled by the remainder $l \equiv n \mod s$, and their asymptotics can be studied via the sequences of reduced polynomials

$$\{q_k^{(l)}(u)\}_{k=0}^{\infty}\ , \qquad l = 0, 1, \dots, s-1.$$

By a simple change of coordinate it is easy to see that the monic polynomials in the sequence $\{q_k^{(l)}\}_{k=0}^{\infty}$ are orthogonal with respect to the measure

$$|u|^{-2\gamma} e^{-N(|u|^2 - tu - t\bar{u})} dA(u), \quad \gamma := \frac{s-l-1}{s} \in [0,1), \tag{2.6}$$

namely they satisfy the orthogonality relations

$$\int_{\mathbb{C}} q_k^{(l)}(u) \bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)} dA(u) = 0, \qquad j = 0, \dots, k-1. \tag{2.7}$$

As a result of this symmetry reduction, starting from the class of measures (2.1), it is sufficient to consider the orthogonal polynomials with respect to the family of measures (2.6).

Remark. *It is clear from the above relation that for $l = s-1$ one has $\gamma = 0$ and the polynomials $q_k^{(s-1)}(u)$ are polynomials of the form*

$$q_k^{(s-1)}(u) = (u-t)^k.$$

It follows that the monic polynomials $p_{ks+s-1}(\lambda)$ have the form

$$p_{ks+s-1}(\lambda) = \lambda^{s-1}(\lambda^s - t)^k.$$

Remark. *Observe that the weight in the orthogonality relation* (2.7) *can be written in the form*

$$|u|^{-2\gamma} e^{-N(|u|^2 - tu - t\bar{u})} = e^{-N(|u-t|^2 + t^2 + 2\frac{\gamma}{N}\log|u|)},$$

and it is similar to the weight $e^{-N\mathcal{V}(u)}$ with $\mathcal{V}(u) = |u|^2 - c\log|u-a|$ with $c > 0$ studied in [2]. However in our case it turns out that $c = -2\gamma/N < 0$, so the point interaction near $a = 0$ is repulsive and the asymptotic distribution of the zeroes of the polynomials (2.7) *turns out to be substantially different from the one in [2].*

In the following we are going to study the asymptotic behaviour of the orthogonal polynomials $p_n(\lambda)$. Since such polynomials can be divided in s different families of polynomials $q_k^l(\lambda^d)$, $l = 0, \dots, l-1$ as introduced in (2.5), we are going to study the asymptotic behaviour of the polynomials $q_k^{(l)}(u)$ with orthogonality relation

$$\int_{\mathbb{C}} q_k^{(l)}(u) \bar{u}^j |u|^{\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)} dA(u) = 0, \qquad j = 0, \dots, k-1. \tag{2.8}$$

The crucial step in the present analysis is to replace the two-dimensional integral conditions (2.8) by an equivalent set of linear constraints in terms of contour integrals.

The change of coordinate

$$u = -t(z-1), \quad z \in \mathbb{C},$$

leads to the following equivalent characterization of $q_k(u)$:

Theorem 2.1. *Let $q_k^{(l)}(u)$ be the monic orthogonal polynomial of degree k characterized by the conditions*

$$\int_{\mathbb{C}} q_k(u)\bar{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\bar{u}\right)}dA(u) \quad j=0,1,\dots,k-1\ ,$$

where $\gamma\in[0,1)$, $N>0$ and $t\in\mathbb{C}$. The transformed monic polynomial

$$\pi_k(z):=\frac{(-1)^k}{t^k}q_k(-t(z-1))$$

of degree k is characterized by the non-hermitian orthogonality relations

$$\oint_\Sigma \pi_k(z)z^j\frac{e^{-Nt^2z}}{z^k}\left(\frac{z}{z-1}\right)^\gamma dz=0 \qquad j=0,1,\dots,k-1\ , \tag{2.9}$$

where Σ is a simple positively oriented contour encircling $z=0$ and $z=1$.

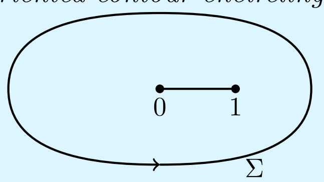


Figure 2.1: The integration contour Σ

Proof. In order to prove the theorem we first show that the orthogonality relation for the polynomials $q_k^l(u)$ on the plane can be reduced to an orthogonality relation on a contour. Let us look for a function $\chi_j(u,\bar{u})$ such that solves the following $\bar{\partial}$-problem

$$\partial_{\overline{u}}\chi_j(u,\bar{u})=\overline{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\overline{u}\right)}. \tag{2.10}$$

Having such a function, for any polynomial $q(u)$ one has

$$d\left[q(u)\chi_j(u,\overline{u})du\right]=q(u)\partial_{\overline{u}}\chi_j(u,\overline{u})d\overline{u}\wedge du=q(u)\overline{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\overline{u}\right)}d\overline{u}\wedge du,$$

where d denote the operation of differentiation, namely $df(u,\bar{u})=\dfrac{\partial f}{\partial u}du+\dfrac{\partial f}{\partial\bar{u}}d\bar{u}$. If such function exists, one can use Stokes' theorem and reduce the orthogonality relation on the plane to an orthogonality relation on a suitable contour. The equation (2.10) has a contour integral solution

$$\begin{aligned}
\chi_j(u,\bar{u}) &= u^{-\gamma}e^{Ntu}\int_0^{\overline{u}} s^{j-\gamma}e^{-Nus+Nts}ds\\
&=\frac{1}{N^{j-\gamma+1}}\left(1-\frac{t}{u}\right)^\gamma\frac{e^{Ntu}}{(u-t)^j}\int_0^{N\overline{u}(u-t)} r^{j-\gamma}e^{-r}dr\\
&=\frac{1}{N^{j-\gamma+1}}\left(1-\frac{t}{u}\right)^\gamma\frac{e^{Ntu}}{(u-t)^j}\left[\Gamma(j-\gamma+1)-\int_{N\overline{u}(u-t)}^\infty r^{j-\gamma}e^{-r}dr\right]\\
&=\frac{\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\left(1-\frac{t}{u}\right)^\gamma\frac{e^{Ntu}}{(u-t)^j}\left[1-\mathcal{O}\left(e^{-N\overline{u}(u-t)}\right)\right] \quad |u|\to\infty
\end{aligned}$$

It follows that for any polynomial $q(u)$ the following integral identity holds:

$$\begin{aligned}
\int_{\mathbb{C}} q(u)\bar{u}^j|u|^{-2\gamma}e^{-N(|u|^2-tu-t\overline{u})}dA(u) &= \lim_{R\to\infty}\int_{|u|\le R} q(u)\bar{u}^j|u|^{-2\gamma}e^{-N(|u|^2-tu-t\overline{u})}dA(u) \\
&= \frac{1}{2i}\lim_{R\to\infty}\int_{|u|\le R} q(u)\bar{u}^j|u|^{-2\gamma}e^{-N(|u|^2-tu-t\overline{u})}d\overline{u}\wedge du \\
&= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\chi_j(u,\overline{u})du \\
&= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\left[G_j(u)-\mathcal{O}\left(e^{-\overline{u}(u-t)}\right)\right]du \\
&= \frac{1}{2i}\oint_{|z|=R_0} q(u)G_j(u)du
\end{aligned}$$

where R and R_0 are sufficiently large and

$$G_j(u) = \frac{\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\left(1-\frac{t}{u}\right)^{\gamma}\frac{e^{Ntu}}{(u-t)^j}\ .$$

So it follows that for any polynomial $q(u)$ the following identity is satisfied:

$$\int_{\mathbb{C}} q(u)\bar{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\bar{u}\right)}dA(u) = \frac{\pi\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\frac{1}{2\pi i}\oint_{\tilde{\Sigma}} q(u)\frac{e^{Ntu}}{(u-)^{j+1}}\left(1-\frac{t}{u}\right)^{\gamma}du\ ,$$

where $\gamma\in[0,1)$, j is an arbitrary non-negative integer, and $\tilde{\Sigma}$ is a positively oriented simple closed loop enclosing $u=0$ and $u=t$. Making the change of coordinate $u=-t(z-1)$ one arrives at the statement of the theorem. □

Let us introduce the weight

$$w_k(z) := e^{-kV(z)}\left(1-\frac{1}{z}\right)^{-\gamma}\ ,$$

where

$$V(z) = \frac{z}{z_0}+\log(z),\quad z_0 := \frac{t_c^2}{t^2}. \tag{2.11}$$

Then, with the choice $N=\dfrac{n-l}{T}$, the orthogonality relations (2.9) can be written in the form

$$\oint_{\Sigma}\pi_k(z)z^jw_k(z)\,dz = 0 \qquad j=0,1,\ldots,k-1$$

We first want to write the polynomials $\pi_k(z)$ as the solution of a Riemann-Hilbert problem. Let us first define

$$\nu_j := \oint_{\Sigma} z^jw_k(z)\mathrm{d}z.$$

Introduce the polynomial

$$\Pi_{k-1}(z) := \frac{1}{\det\left[\nu_{i+j}\right]_{0\le i,j\le k-1}}\det\begin{bmatrix}\nu_0 & \nu_1 & \ldots & \nu_{k-1}\\ \nu_1 & \nu_2 & \ldots & \nu_k\\ \vdots & & & \vdots\\ \nu_{k-2} & \cdots & & \nu_{2k-3}\\ 1 & z & \ldots & z^{k-1}\end{bmatrix} \tag{2.12}$$

Note that Π_{k-1} is not necessarily monic and its degree may be less than $k-1$: its existence is guaranteed just by requiring that the determinant in the denominator does not vanish.

Proposition 2.1. *The determinant* $\det[\nu_{i+j}]_{0\le i,j\le k-1}$ *does not vanish.*

Proof. We have

$$\begin{aligned}\det[\nu_{i+j}]_{0\le i,j\le k-1} &= \det\left[\oint_\Sigma z^{i+j}\frac{e^{-Nt^2z}}{z^k}\left(\frac{z}{z-1}\right)^\gamma dz\right]_{0\le i,j\le k-1}\\ &= (-1)^{k(k-1)/2}\det\left[\oint_\Sigma z^{i-j}\frac{e^{-Nt^2z}}{z}\left(\frac{z}{z-1}\right)^\gamma dz\right]_{0\le i,j\le k-1},\end{aligned}$$

where the last identity has been obtained by the reflection of the column index $j\to k-1-j$. Due to Theorem 2.1 we have

$$\int_{\mathbb{C}}\pi(z)(\bar z-1)^j|z-1|^{-2\gamma}e^{-Nt^2|z|^2}dA(z) = t^{2-2j-2\gamma}\frac{\pi\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\frac{1}{2\pi i}\oint_\Sigma \pi(z)\frac{e^{-Nt^2z}}{z^{j+1}}\left(\frac{z}{z-1}\right)^\gamma dz\,,$$

an hence the second determinant is given by

$$\begin{aligned}&\det\left[\oint_\Sigma z^{i-j}\frac{e^{-Nt^2z}}{z}\left(\frac{z}{z-1}\right)^\gamma dz\right]\\ &=\det\left[\iint_{\mathbb{C}} z^i(\bar z-1)^j|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\mathrm{d}A(z)\right]\prod_{j=0}^{k-1}\frac{2it^{2j+2\gamma-2}N^{j-\gamma+1}}{\Gamma(j-\gamma+1)}.\end{aligned}\tag{2.13}$$

Finally, the determinant on the right-hand side is strictly positive because

$$\det\left[\iint_{\mathbb{C}} z^i(\bar z-1)^j|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\mathrm{d}A(z)\right] = \det\left[\iint_{\mathbb{C}} z^i\bar z^j|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\mathrm{d}A(z)\right]>0$$

where the equality follows from the fact that the columns of the two matrices are related by a unimodular triangular matrix, while the inequality follows from the positivity of the measure. Finally, since $\Gamma(z)$ has no zeroes (and no poles since $j-\gamma+1>0$), the non-vanishing follows from (2.13). □

2.2 Riemann–Hilbert problem

Define the matrix

$$Y(z)=\begin{bmatrix}\pi_k(z) & \dfrac{1}{2\pi i}\displaystyle\int_\Sigma\frac{\pi_k(z')}{z'-z}w_k(z')dz'\\ -2\pi i\Pi_{k-1}(z) & -\displaystyle\int_\Sigma\frac{\Pi_{k-1}(z')}{z'-z}w_k(z')dz'\end{bmatrix}.\tag{2.14}$$

Then the matrix $Y(z)$ satisfies the standard Riemann-Hilbert problem for orthogonal polynomials [16]

Fokas-Its-Kitaev Riemann-Hilbert problem. For a 2×2 matrix $Y(z)$

1. The matrix

$$Y(z) \text{is analytic in } \mathbb{C} \setminus \Sigma\,, \tag{2.15}$$

2. the jump on Σ

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_k(z) \\ 0 & 1 \end{pmatrix}\,, \tag{2.16}$$

3. behaviour at infinity:

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{k\sigma_3}\,, \quad z \to \infty. \tag{2.17}$$

The last relation is obtained by noticing that if Π_k is given by (2.12) then the entry 22 of the matrix $Y(z)$ in (2.14) satisfies

$$-\int_\Sigma \frac{\Pi_k(z')}{z'-z} w_k(z') dz' \sim z^{-k}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)\,.$$

2.2.1 Undressing

In order to simplify the analysis, we define the following matrix:

$$\tilde{Y}(z) := Y(z)\left(1 - \frac{1}{z}\right)^{-\frac{\gamma}{2}\sigma_3} \qquad z \in \mathbb{C} \setminus (\Sigma \cup [0,1])\,. \tag{2.18}$$

This matrix-valued function $\tilde{Y}(z)$ satisfies the following Riemann-Hilbert problem:

1. $\tilde{Y}(z)$ is analytic in $\mathbb{C} \setminus (\Sigma \cup [0,1])$

2. the jump on Σ:

$$\tilde{Y}_+(z) = \tilde{Y}_-(z) \begin{pmatrix} 1 & e^{-kV(z)} \\ 0 & 1 \end{pmatrix}\,,$$

   the jump on $(0,1)$:

$$\tilde{Y}_+(z) = \tilde{Y}_-(z) e^{-\gamma\pi i\sigma_3}\,.$$

3. large z boundary behaviour:

$$\tilde{Y}(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{k\sigma_3}\,, \quad z \to \infty\,.$$

4. endpoint behaviour:

$$\tilde{Y}(z) z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0\,, \qquad \tilde{Y}(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1\,.$$

The polynomials $\pi_k(z)$ is recovered from

$$\pi_k(z) = \tilde{Y}_{11}(z)\left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}}.$$

Our aim is to study the behaviour of the polynomials π_k in the limit $k \to \infty$ and $N \to \infty$ is such a way that for $n = kd + l$ one has

$$N = T(n - l), \quad T > 0.$$

With this definition it turns out that the potential $V_k(z)$ define in (2.11) is independent from k and equal to[2]

$$V(z) = \log z + \frac{z}{z_0}, \quad z_0 = \frac{t_c^2}{t^2}, \quad t_c^2 = \frac{T}{s}$$

In the limit $k \to \infty$, we distinguish two different cases:

- pre-critical case $z_0 > 1$,
- post-critical case $z_0 < 1$.

2.3 Asymptotic analysis: pre-critical case

In order to analyse the large k behaviour of $\tilde{Y}$ we use the Deift-Zhou steepest descent method [9]. The first step to study the large k behaviour of the matrix function $\tilde{Y}$ is to make a transformation $\tilde{Y}(z) \to U(z)$ so that the Riemann-Hilbert problem for $U(z)$ is normalised to the identity as $|z| \to \infty$. For the purpose we introduce a contour Γ homotopically equivalent to Σ in $\mathbb{C} \setminus [0,1]$ and a function $g(z)$ analytic off Γ. Both the contour Γ and the function $g(z)$ are unknown. We assume that the function $g(z)$ is of the form

$$g(z) = \int_\Gamma \log(z - s)d\nu(s) \tag{2.19}$$

where $d\nu(s)$ is a positive measure with support on Γ such that

$$\int_\Gamma d\nu(s) = 1.$$

With this assumption clearly one has

$$g(z) = \log z + O(z^{-1}) \quad \text{az } |z| \to \infty. \tag{2.20}$$

2.3.1 First transformation $\tilde{Y} \to U$

Since Γ is homotopically equivalent to Σ in $\mathbb{C}\setminus[0,1]$ we can deform the contour Σ appearing in the Riemann–Hilbert problem for $\tilde{Y}$ to unknown contour Γ homotopic to Σ. Define the modified matrix

$$U(z) = e^{-k(\ell/2)\sigma_3}\tilde{Y}(z)e^{-kg(z)\sigma_3}e^{k(\ell/2)\sigma_3} \qquad z \in \mathbb{C} \setminus (\Gamma \cup [0,1]),$$

where ℓ is a real number. Then $U(z)$ solves the following RHP problem

[2] Notice that the potential $V(z)$ defined here in **not** the same potential defined in (2.1).

- $U(z)$ is analytic in $\mathbb{C} \setminus (\Gamma \cup [0,1])$.
- The jump on Γ:

$$U_+(z) = U_-(z) \begin{pmatrix} e^{-k(g_+-g_-)} & e^{k(g_++g_--\ell-V)} \\ 0 & e^{k(g_+-g_-)} \end{pmatrix} , \tag{2.21}$$

- The jump on $(0,1)$:

$$U_+(z) = U_-(z)e^{-\gamma\pi i\sigma_3} ,$$

- Endpoint behaviour:

$$U(z)z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0 , \qquad U(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1 .$$

- Large z boundary behaviour:

$$U(z) = I + \mathcal{O}\left(\frac{1}{z}\right) , \quad z \to \infty . \tag{2.22}$$

The polynomials $\pi_k(z)$ is determined from $U(z)$ by

$$\pi_k(z) = U_{11}(z)e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}} . \tag{2.23}$$

In order to determine the function g and the contour Γ we impose that the jump matrix (2.21) becomes purely oscillatory for large k. This is accomplished if the following conditions are satisfied

$$\begin{aligned} &g_+(z) + g_-(z) - \ell - V(z) = 0, \quad z \in \Gamma \\ &\mathrm{Re}(g_+(z) - g_-(z)) = 0, \quad z \in \Gamma. \end{aligned} \tag{2.24}$$

Next we show that we can find a function g and a contour Γ that satisfy the conditions (2.24). For the purpose we use the following elementary result in the theory of boundary value problems.

Lemma 2.1 ([17], p. 78)**.** *Let L be a simple closed contour dividing the complex plane in two regions D_+ and D_-. Suppose that a function $\psi(\zeta)$ defined on L can be represented in the form*

$$\psi(\zeta) = \psi_+(\zeta) + \psi_-(\zeta), \quad \zeta \in L$$

where $\psi_+(\zeta)$ is the boundary value of a function $\psi_+(z)$ analytic for $z \in D_+$ and $\psi_-(\zeta)$ is the boundary value of a function $\psi_-(z)$ analytic for $z \in D_-$ and such that $\psi_-(\infty) = 0$. Then the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\psi(\zeta)}{\zeta - z}d\zeta$$

can be represented in the form

$$\begin{aligned} \Phi_+(z) &= \psi_+(z), \quad \text{for } z \in D_+, \\ \Phi_-(z) &= -\psi_-(z), \ \text{for } z \in D_-. \end{aligned}$$

The boundary values of the function Φ on the two sides of the contour L then satisfy

$$\begin{aligned}\Phi_+(\zeta)+\Phi_-(\zeta)&=\psi_+(\zeta)-\psi_-(\zeta), \quad \zeta\in L\\ \Phi_+(\zeta)-\Phi_-(\zeta)&=\psi(\zeta)=\psi_+(\zeta)+\psi_-(\zeta), \quad \zeta\in L.\end{aligned}$$

We are going to use the above lemma for the function $g'(z)$ that satisfies the boundary condition

$$g'_+(\zeta)+g'_-(\zeta)=V'(\zeta)=\frac{1}{z_0}+\frac{1}{\zeta}, \quad \zeta\in\Gamma.$$

In this case the functions $\psi_\pm(z)$ of Lemma 2.1 are $\psi_+(z)=\dfrac{1}{z_0}$ and $\psi_-(z)=-\dfrac{1}{z}$ and therefore the function $g'(z)$ takes the form

$$g'(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\frac{1}{\tau}-\frac{1}{z_0}}{z-\tau}d\tau=\begin{cases}\dfrac{1}{z_0} & z\in\mathrm{int}(\Gamma)\\[2ex] \dfrac{1}{z} & z\in\mathrm{ext}(\Gamma)\,,\end{cases} \tag{2.25}$$

so that the measure $d\nu$ in (2.19) is given by

$$d\nu(z)=\frac{1}{2\pi i}\left(\frac{1}{z}-\frac{1}{z_0}\right)dz, \quad z\in\Gamma. \tag{2.26}$$

Integrating the relation (2.25) and using (2.20) one has

$$g(z)=\begin{cases}\dfrac{z}{z_0}+\ell & z\in\mathrm{int}(\Gamma)\\[2ex] \log z & z\in\mathrm{ext}(\Gamma)\end{cases} \tag{2.27}$$

where ℓ is an integration constant and $\log z$ is analytic in $\mathbb{C}\backslash\mathbb{R}^+$. Performing the integral in (2.19) for a specific value of $z\in\Gamma$, say $z=0$ and deforming Γ to a circle of radius r one can determine ℓ which turns out to be given by

$$\ell=\log r-\frac{r}{z_0}, \quad r>0.$$

The quantity $d\nu$ in (2.26) is normalised to one on any close contour containing the point $z=0$. However we have to define a contour Γ in such a way that $d\nu(z)$ is a real and positive measure on such contour. For the purpose, let us introduce the function

$$\phi_r(z)=\begin{cases}-2g(z)+V(z)+\log r-\dfrac{r}{z_0}=\log z-\dfrac{z}{z_0}-\log r+\dfrac{r}{z_0} & z\in\mathrm{int}(\Gamma\backslash[0,r])\\[2ex] 2g(z)-V(z)-\log r+\dfrac{r}{z_0}=\log z-\dfrac{z}{z_0}-\log r+\dfrac{r}{z_0} & z\in\mathrm{ext}(\Gamma\backslash(r,\infty))\,.\end{cases} \tag{2.28}$$

Oberve that $\phi_r(z)$ is analytic through Γ, namely $\phi_{r+}(z)=\phi_{r-}(z)$, $z\in\Gamma$ and that

$$\phi_r(z)=\frac{\phi_{r+}(z)+\phi_{r-}(z)}{2}=-g_+(z)+g_-(z), \quad z\in\Gamma. \tag{2.29}$$

The next identity follows in a trivial way

$$d\nu(z)=\frac{1}{2\pi i}d\phi_r(z).$$

Imposing the relation (2.24) on (2.29) one has

$$\mathrm{Re}(\phi_r(z)) = \log|z| - \frac{\mathrm{Re}(z)}{z_0} - \log r + \frac{r}{z_0} = 0.$$

Such equation defines a family of contours which are closed for $|z| \le r$. Since the function $-\log r + \frac{r}{z_0}$ has a single minimum at $r = z_0$ it is sufficient to consider the values $0 < r \le z_0$. We define the contour Γ_r as (see Figure 2.2)

$$\Gamma_r = \{z \in \mathbb{C},\ s.t.\ \mathrm{Re}(\phi_r(z)) = 0,\ \ |z| \le z_0\}, \quad 0 < r \le z_0. \tag{2.30}$$

We observe that the point $z = 0$ lies inside Γ_r because $\mathrm{Re}\ \phi(z) \to -\infty$ for $|z| \to 0$. Furthermore Γ_r intersect the real line in the points $z = r$ and $z > -r$. Indeed $\mathrm{Re}\ \phi(-r) > 0$ which shows that the point $z = -r$ lies outside Γ_r.

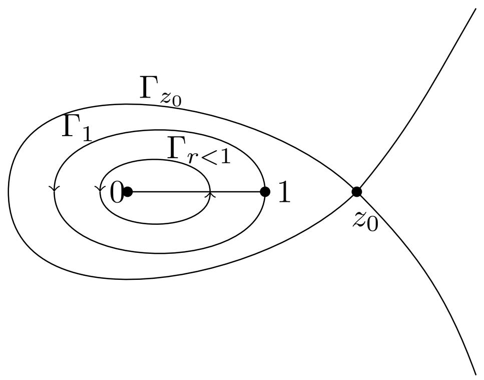


Figure 2.2: The family of contours Γ_r for three values of r: $r = z_0$, $r = 1$ and $r < 1$. For $r = z_0$, the region $|z| > z_0$ is plotted as well.

Lemma 2.2. *The a-priori complex measure ν in* (2.26) *is a unit positive measure on the contour Γ_r defined in* (2.30) *for $0 < r \le z_0$.*

Proof. Applying the residue theorem it is clear that the measure $d\nu$ is normalised to one on any contour Γ_r. Introducing the variable

$$\psi_r = e^{\phi_r},$$

we can re-write the measure $d\nu$ in the form

$$d\nu = \frac{1}{2\pi i} d\phi_r = \frac{1}{2\pi i}\frac{d\psi_r}{\psi_r}.$$

On the contour Γ_r we have that

$$|\psi_r| = 1,$$

so that in the ψ_r-plane the contour Γ_r is a circle of radius one and the map $\psi_r = e^{\phi_r}$ is a univalent conformal map from the interior of Γ_r to the interior of a circle. Parametrizing the unit circle as $e^{i\theta}$ with $0 \le \theta \le 2\pi$, we have that

$$d\nu = \frac{1}{2\pi i}\frac{d\psi_r}{\psi_r} = \frac{1}{2\pi} d\theta, \quad 0 \le \theta \le 2\pi.$$

Namely in the variable ψ_r the measure $d\nu$ is a uniform measure on the circle and this shows that the measure $d\nu$ is a positive measure on each contour Γ_r. $\square$

2.3.2 Choice of the contour

So far we can deform our original contour Σ to a whole family of contours Γ_r, $0 < r \leq z_0$. We now argue that the relevant contour on which the zeroes of the orthogonal polynomials $\pi_k(z)$ accumulate is given by the level $r = 1$. The family of contours $0 < r < 1$ can be immediately ignored because in this case Σ has to be deformed to two contours $\Sigma \simeq \Gamma_{r<1} \cup \tilde{\Gamma}$ as shown in the Figure 2.3.

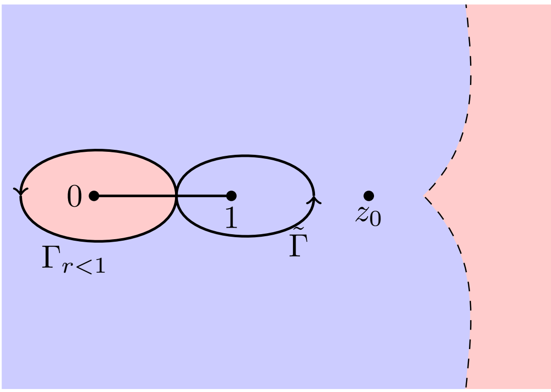


Figure 2.3: The contour $\Gamma_{r<1} \cup \tilde{\Gamma}$ that is homotopic to Σ. The region where $\operatorname{Re}\phi(z) < 0$ is coloured in red.

On the contour $\tilde{\Gamma}$ we have $\operatorname{Re}(\phi) > 0$ and it is not possible to perform any contour deformation to get exponential small terms in the jump matrix (2.21) as $k \to \infty$.

In the case $1 < r < z_0$ the asymptotic analysis gives a trivial RH-problem, with no sub-leading corrections as $k \to \infty$. The only possible cases are

- $r = 1$,
- $r = z_0$.

In the case $r = 1$ the error matrix $R(z)$ (see below Section 2.3.5) is upper triangular, giving a nonzero contribution in the asymptotic analysis as $k \to \infty$ to the entry of the matrix $Y_{11}(z) = \pi_k(z)$. Therefore in this case, having a leading and sub-leading term in the asymptotic expansion of the orthogonal polynomials $\pi_k(z)$ we are able to locate their zeroes on a contour that lies within a distance $\log k/k$ from the contour $\Gamma_{r=1}$.

In the case $r = z_0$ the error matrix $R(z)$ is lower triangular, giving a nonzero contribution in the asymptotic analysis as $k \to \infty$ to the entry of the matrix $Y_{21}(z)$ only. In this case, the only zero of the polynomials $\pi_k(z)$ we are able to locate are $z = 0$ and $z = 1$.

So for the reasons explained above, we are going to perform the asymptotic analysis of the RH-problem (2.21)-(2.22) deforming the contour Σ to the contour $\Gamma_{r=1}$. For simplicity we denote this contour Γ:

$$\Gamma = \{z \in \mathbb{C},\ s.t.\ \log|z| - \frac{\operatorname{Re}(z)}{z_0} + \frac{1}{z_0} = 0,\ \ |z| \leq 1\}$$

and the function $\phi_{r=1}$ as ϕ:

$$\phi(z) = \log z - \frac{z}{z_0} + \frac{1}{z_0}. \tag{2.31}$$

With this choice of contour, the jump matrices for the RH problem (2.21)-(2.22) of the matrix $U(z)$ can be summarised in the figure below.

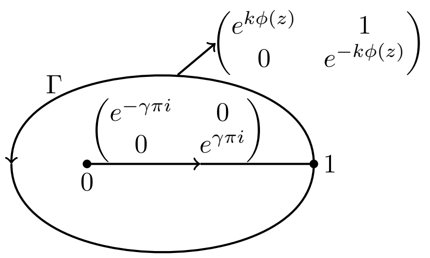


Figure 2.4: The jump matrices for $U(z)$

2.3.3 The second transformation $U \mapsto T$

Consider two extra loops Γ_i and Γ_e as shown in Figure 2.5. These define new domains Ω_0, Ω_1, Ω_2 and Ω_∞. Define

$$T(z) = \begin{cases} U(z) & z \in \Omega_\infty \cup \Omega_0 \\ U(z) \begin{pmatrix} 1 & 0 \\ -e^{k\phi(z)} & 1 \end{pmatrix} & z \in \Omega_1 \\ U(z) \begin{pmatrix} 1 & 0 \\ e^{-k\phi(z)} & 1 \end{pmatrix} & z \in \Omega_2 \,. \end{cases} \tag{2.32}$$

Then this matrix-valued function has the following jump discontinuities:

$$T_+(z) = T_-(z) v_T(z), \quad z \in \Sigma_T,$$

where Σ_T is the contour defined in the Figure 2.5 and the matrix $v_T(z)$ takes the form

$$v_T(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in \Gamma \\ \begin{pmatrix} 1 & 0 \\ e^{-k\phi(z)} & 1 \end{pmatrix} & z \in \Gamma_e \\ \begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix} & z \in \Gamma_i \\ e^{-\gamma\pi i \sigma_3} & z \in (0,1) \,. \end{cases}$$

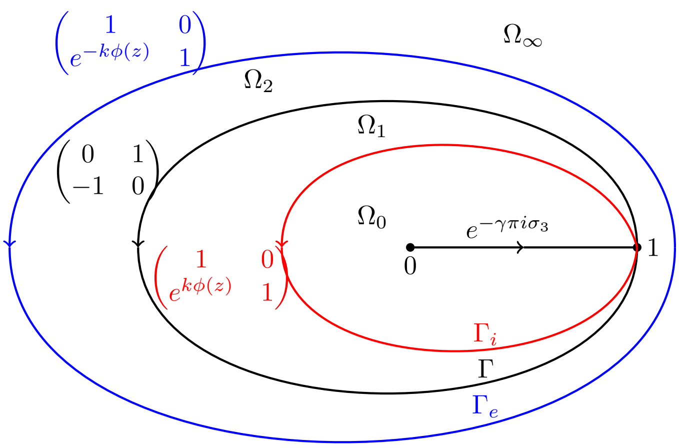


Figure 2.5: The jump matrices for $T(z)$ with contour $\Sigma_T = \Gamma_e \cup \Gamma \cup \Gamma_i \cup (0,1)$.

Endpoint behaviour:

$$T(z)z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0 \, , \qquad T(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1 \, .$$

Large z boundary behaviour:

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \, , \quad z \to \infty \, .$$

Note that $e^{k\phi(z)}$ is analytic in $\mathbb{C} \setminus \{0\}$. The important feature of this RHP is that the jumps are either constant or exponentially decaying in k (though not uniformly).

Proposition 2.2. *There exists a constant $c_0 > 0$ so that*

$$v_T(z) = v^\infty(z)(I + e^{-c_0 k}) \quad \text{as } k \to \infty$$

uniformly for $z \in \Sigma_T \backslash \mathcal{U}_1$, where $\mathcal{U}_1$ is a small neighbourhood of 1 and

$$v^\infty(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{as } z \in \Gamma, \\ \\ e^{-\gamma\pi i\sigma_3} & \text{as } z \in (0,1). \end{cases} \tag{2.33}$$

2.3.4 The outer parametrix

We need to find a matrix-valued function $P^\infty(z)$ analytic in $\mathbb{C} \setminus (\Gamma \cup [0,1])$ such that it has jump discontinuities given by the matrix $v^\infty(z)$ in (2.33), namely

$$P^\infty_+(z) = P^\infty_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad z \in \Gamma \, ,$$

and

$$P^\infty_+(z) = P^\infty_-(z) e^{-\gamma\pi i\sigma_3} \qquad z \in (0,1) \, ,$$

with endpoint behaviour:

$$P^{\infty}(z)z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0 \ , \qquad P^{\infty}(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1 \ ,$$

and large z boundary behaviour:

$$P^{\infty}(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \ , \quad z \to \infty \ .$$

Define

$$\tilde{P}^{\infty}(z) := P^{\infty}(z)\chi^{-1}(z)$$

where

$$\chi(z) := \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in \mathrm{int}(\Gamma) \\ I & z \in \mathrm{ext}(\Gamma) \ . \end{cases} \tag{2.34}$$

The k-independent matrix $\tilde{P}^{\infty}(z)$ has no jump on Γ and it satisfies the following RHP:

- $\tilde{P}^{\infty}$ is holomorphic in $\mathbb{C} \setminus (0, 1)$
- jump across $(0, 1)$:
$$\tilde{P}^{\infty}_{+}(z) = \tilde{P}^{\infty}_{-}(z)e^{\gamma\pi i\sigma_3}$$
- large z boundary behaviour:
$$\tilde{P}^{\infty}(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \ , \quad z \to \infty \ .$$

A particular solution is given by

$$\tilde{P}^{\infty}(z) = \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3}$$

which leads to the particular solution

$$P^{\infty}(z) = \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3}\chi(z) = \begin{cases} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in \mathrm{int}(\Gamma) \\ \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3} & z \in \mathrm{ext}(\Gamma) \ . \end{cases} \tag{2.35}$$

2.3.5 The local parametrix at $z = 1$

The aim of this section is to construct a local parametrix $P^{0}(z)$ in a small neighborhood $\mathcal{U}_1$ of $z = 1$ having the same jump property as T for z near 1 and matching the outer parametric $P^{\infty}(z)$ in the limit $k \to \infty$ and $z \in \partial\mathcal{U}_1$. Then the Riemann-Hilbert problem for $P^{0}(z)$ is given by

$$P^{0}_{+}(z) = P^{0}_{-}(z)v_T(z), \quad z \in \mathcal{U}_1 \cap \Sigma_T,$$

and

$$P^0(z) = P^\infty(z)(I + o(1)) \qquad \text{as } k \to \infty \text{ and } z \in \partial\mathcal{U}_1. \tag{2.36}$$

In order to build such local parametrix near the point $z = 1$ we first construct a new matrix function $V(z)$ from $P^0(z)$:

$$V(z) = P^0(z)\chi(z)^{-1}\mathcal{Q}(z)$$

where

$$\mathcal{Q}(z) := \begin{cases} \begin{pmatrix} 1 & e^{k\phi(z)} \\ 0 & 1 \end{pmatrix} & z \in \Omega_0 \cap \mathcal{U}_1 \\ I & z \in \mathcal{U}_1 \setminus \Omega_0 \end{cases} \tag{2.37}$$

The matrix $V(z)$ satisfies the following jump relations in a neighbourhood of 1:

$$V_+(z) = V_-(z) \begin{pmatrix} e^{\gamma\pi i} & (e^{\gamma\pi i} - e^{-\gamma\pi i})e^{k\phi(z)} \\ 0 & e^{-\gamma\pi i} \end{pmatrix} \qquad z \in \mathbb{R}^- .$$

Model problem

Consider the model problem for the 2×2 matrix function $\Psi(\xi)$ analytic in $\mathbb{C}\backslash\mathbb{R}^-$ with boundary behaviour

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} e^{\gamma\pi i} & (e^{\gamma\pi i} - e^{-\gamma\pi i})e^{\xi} \\ 0 & e^{-\gamma\pi i} \end{pmatrix} \qquad \xi \in \mathbb{R}^-$$

$$\Psi(\xi) = \left(I + \mathcal{O}\left(\frac{1}{\xi}\right)\right) \xi^{\frac{\gamma}{2}\sigma_3} \qquad \xi \to \infty .$$

Defining

$$\tilde{\Psi}(\xi) := \Psi(\xi)\xi^{-\frac{\gamma}{2}\sigma_3}$$

we obtain the following Riemann–Hilbert problem for $\tilde{\Psi}(\xi)$:

$$\tilde{\Psi}_+(\xi) = \tilde{\Psi}_-(\xi) \begin{pmatrix} 1 & (1 - e^{-2\gamma\pi i})(\xi^\gamma)_+ e^{\xi} \\ 0 & 1 \end{pmatrix} \qquad \xi \in \mathbb{R}^-$$

$$\tilde{\Psi}(\xi) = \left(I + \mathcal{O}\left(\frac{1}{\xi}\right)\right) \qquad \xi \to \infty .$$

This is an abelian Riemann–Hilbert problem, and therefore easily solvable by the Sokhotski–Plemelj formula:

$$\tilde{\Psi}(\xi) = \begin{pmatrix} 1 & -\dfrac{1 - e^{-2\gamma\pi i}}{2\pi i} \displaystyle\int_{\mathbb{R}^-} \frac{(\zeta^\gamma)_+ e^{\zeta} d\zeta}{\zeta - \xi} \\ 0 & 1 \end{pmatrix}, \tag{2.38}$$

so that

$$\Psi(\xi) = \left[I + \sum_{j=1}^{\infty} \frac{\Psi_j}{\xi^j}\Lambda\right] \xi^{\frac{\gamma}{2}\sigma_3}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{as } \xi \to \infty. \tag{2.39}$$

In particular

$$\Psi_1 = \frac{e^{-2\gamma\pi i} - 1}{2\pi i} \int_{\mathbb{R}^-} (\zeta^\gamma)_+ e^{\zeta} d\zeta = \frac{e^{-\gamma\pi i} - e^{\gamma\pi i}}{2\pi i} \int_0^\infty r^\gamma e^{-r} dr = -\frac{\sin(\gamma\pi)}{\pi}\Gamma(\gamma+1) = \frac{1}{\Gamma(-\gamma)} . \tag{2.40}$$

Construction of the parametrix

We are now ready to specify the parametric $P^0(z)$ which is of the following form

$$P^0(z) = E(z)\Psi(kw(z))\mathcal{Q}(z)^{-1}\chi(z)\,,$$

where $\mathcal{Q}(z)$ and $\chi(z)$ are defined in (2.62)) and (2.34) respectively and $E(z)$ is an analytic matrix in a neighbourhood of $\mathcal{U}_1$ and $w(z)$ is a conformal mapping from a neighbourhood of 1 to a neighbourhood of 0.

The conformal map $w(z)$ is specified by

$$w(z) := \begin{cases} \phi(z) + 2\pi i & z \in \mathcal{U}_1 \cap \mathbb{C}_- \\ \phi(z) & z \in \mathcal{U}_1 \cap \mathbb{C}_+\ . \end{cases}$$

We observe that

$$w(z) = \left(1 - \frac{1}{z_0}\right)(z-1) - \frac{1}{2}(z-1)^2 + \mathcal{O}\left((z-1)^3\right) \qquad z \to 1\ .$$

The matrix $E(z)$ is obtained from condition (2.36) which, when combined with (2.40) gives

$$\begin{aligned} &P^\infty(z)(P^0(z))^{-1} = P^\infty(z)\chi(z)^{-1}\mathcal{Q}(z)\Psi(kw(z))^{-1}E(z)^{-1} = \\ &= P^\infty(z)\chi(z)^{-1}(kw(z))^{-\frac{\gamma}{2}\sigma_3}\left(I - \frac{1}{\Gamma(-\gamma)}\frac{\Lambda}{kw(z)} + O(k^{-2})\right)E^{-1}(z), \quad k \to \infty, \quad z \in \partial\mathcal{U}_1 \end{aligned} \tag{2.41}$$

where we use the fact that $\mathcal{Q}(z) \to I$ exponentially fast as $k \to \infty$, and $z \in \partial\mathcal{U}_1$. From the above expression it turns out that the matrix $E(z)$ takes the form

$$E(z) = P^\infty(z)\chi(z)^{-1}(kw(z))^{-\frac{\gamma}{2}\sigma_3} = \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3}(kw(z))^{-\frac{\gamma}{2}\sigma_3}. \tag{2.42}$$

We observe that the function $E(z)$ is single valued in a neighbourhood of $\mathcal{U}_1$. Indeed the boundary values of $(z-1)^{\frac{\gamma}{2}\sigma_3}$ and $w^{-\frac{\gamma}{2}\sigma_3}$ cancel each other.

From the expression (2.41) and (2.42) the matching between $P^0(z)$ and $P^\infty(z)$ takes the form

$$\begin{aligned} P^\infty(z)(P^0(z))^{-1} &= E(z)\left(I - \left(\sum_{j=1}^{M-1}\frac{\Psi_j}{(kw)^j} + O(k^{-M})\right)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)E^{-1}(z), \\ &= I - \left(\sum_{j=1}^{M-1}\frac{\Psi_j}{(kw(z))^j} + O(k^{-M})\right)E(z)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}E^{-1}(z) \\ &= I - \left(1 - \frac{1}{z}\right)^{\gamma}(kw(z))^{-\gamma}\left(\sum_{j=1}^{M-1}\frac{\Psi_j}{(kw(z))^j} + O(k^{-M})\right)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \quad k \to \infty, \quad z \in \partial\mathcal{U}_1, \end{aligned} \tag{2.43}$$

where $\gamma \in (0,1)$.

Riemann-Hilbert problem for the error matrix R

We now define the error matrix R in two regions of the plane, using our approximations to the matrix T. Set

$$R(z) = \begin{cases} T(z)\left(P^{(0)}(z)\right)^{-1}, & z \in \mathcal{U}_1 \,, \\ T(z)\left(P^{\infty}(z)\right)^{-1}, & \text{everywhere else.} \end{cases} \tag{2.44}$$

The matrix R is piece-wise analytic in $\mathbb{C}$ with a jump across the contour Γ_R given in Figure 2.6

RH problem for R

(a) R is analytic in $\mathbb{C} \setminus \Gamma_R$,

(b) For $z \in \partial \cup \Gamma_R$, we have

$$R_+(z) = R_-(z) v_R(z), \tag{2.45}$$

with

$$\begin{aligned} v_R(z) &= P^{\infty}_-(z) v_T(z) \left(P^{\infty}_+(z)\right)^{-1}, & z \in \Gamma_R \backslash \mathcal{U}_1 \\ v_R(z) &= P^{\infty}(z) \left(P^{(0)}(z)\right)^{-1}, & z \in \partial\mathcal{U}_1 \end{aligned}$$

(c) As $z \to \infty$, we have

$$R(z) = I + O(z^{-1}).$$

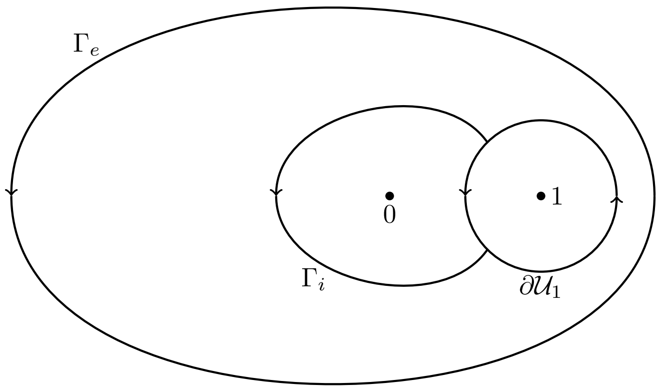


Figure 2.6: The jump contour structure of the matrix R: $\Gamma_R = \Gamma_i \cup \Gamma_e \cup \partial\mathcal{U}_1$.

The jump matrices across the contour $\Gamma_R \setminus \mathcal{U}_1$ are all exponentially close to I for large k because v_T converges exponentially fast to v^{∞} defined in (2.33) and the product $P^{\infty}_-(z) v^{\infty}(z) \left(P^{\infty}_+(z)\right)^{-1} = I$ with $P^{\infty}(z)$ defined in (2.35). The only jump that is not exponentially small is the one on $\partial\mathcal{U}_1$. Indeed one has from (2.43) and any integer $M > 2$,

$$\begin{aligned} v_R(z) &= P^{\infty}(z)(P^0_k(z))^{-1} \\ &= I - \left(1 - \frac{1}{z}\right)^{\gamma} (kw(z))^{-\gamma} \left(\sum_{j=1}^{M-1} \frac{\Psi_j}{(kw(z))^j} + O(k^{-M}) \right) \Lambda. \end{aligned}$$

Defining

$$v_R^{(j)} = -\left(1-\frac{1}{z}\right)^{\gamma}\frac{\Psi_j}{(w(z))^{j+\gamma}}, \quad j=1,2,\dots,$$

we can re-write the matrix $v_R(z)$ in the form

$$v_R = I + \frac{1}{k^\gamma}\left(\sum_{j=1}^{M-1}\frac{v_R^{(j)}}{k^j} + O(k^{-M})\right)\Lambda \tag{2.46}$$

where in particular, using (2.40) one has

$$v_R^{(1)} = -\frac{1}{\Gamma(-\gamma)}\left(1-\frac{1}{z}\right)^{\gamma}(w(z))^{-\gamma-1}. \tag{2.47}$$

By a standard perturbation theory argument one has the expansion

$$R(z) = I + \frac{1}{k^\gamma}\left(\sum_{j=1}^{M-1}\frac{R^{(j)}(z)}{k^j} + O(k^{-M})\right) \quad k\to\infty\ , \tag{2.48}$$

which gives, using (2.45), (2.46) and (2.48)

$$R_+^{(1)}(z) = R_-^{(1)}(z) + v_R^{(1)}(z)\Lambda,$$

with the matrix Λ defined in (2.39). Therefore

$$R^{(1)}(z) = \left[\frac{1}{2\pi i}\oint_{\partial\mathcal{U}_1}\frac{v_R^{(1)}(\zeta)d\zeta}{\zeta - z}\right]\Lambda$$

where we observe that the function $v_R^{(1)}$ has a simple pole in $z=1$ with expansion

$$v_R^{(1)}(z) = -\frac{1}{\Gamma(-\gamma)}\frac{1}{z-1}\left(1-\frac{1}{z_0}\right)^{-\gamma-1}\left(1+\mathcal{O}\left(z-1\right)\right)\ .$$

By a simple residue calculation we obtain

$$R^{(1)}(z) = \begin{cases} \dfrac{1}{(z-1)\Gamma(-\gamma)}\left(1-\dfrac{1}{z_0}\right)^{-\gamma-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & z\in\mathbb{C}\setminus\mathcal{U}_1 \\[2ex] \dfrac{1}{(z-1)\Gamma(-\gamma)}\left(1-\dfrac{1}{z_0}\right)^{-\gamma-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + v_R^{(1)}(z)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & z\in\mathcal{U}_1\ . \end{cases} \tag{2.49}$$

In general, given the structure of the jump matrix (2.46) the error matrix $R^{(j)}(z)$ in (2.48) is of the form

$$R^{(j)}(z) = \begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}, \tag{2.50}$$

namely, only the entry 12 of the matrix $R^{(j)}(z)$ is not zero. From (2.44) and (2.48) one has

$$T(z) = \begin{cases} P^{(0)}(z) + \dfrac{1}{k^\gamma}\left(\displaystyle\sum_{j=1}^{M-1}\frac{R^{(j)}(z)}{k^j} + O(k^{-M})\right)P^{(0)}(z) & z\in\mathcal{U}_1 \\ P^{\infty}(z) + \dfrac{1}{k^\gamma}\left(\displaystyle\sum_{j=1}^{M-1}\frac{R^{(j)}(z)}{k^j} + O(k^{-M})\right)P^{\infty}(z) & \text{everywhere else,} \end{cases} \tag{2.51}$$

where in particular $R^{(1)}(z)$ is given in (2.49).

2.3.6 Asymptotics for $\pi_k(z)$ for $z_0 > 1$

We can now derive the asymptotic expansion of the reduced polynomials $\pi_k(z)$ for $k \to \infty$ with $n = sk + l$. For the purpose we use (2.23), (2.32), (2.49) and (2.51) to obtain

$$
\begin{aligned}
\pi_k(z) &= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}}[U_k(z)]_{11} \\
&= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}}
\begin{cases}
[T_k(z)]_{11} & z \in \Omega_\infty \cup \Omega_0 \\
\left[T_k(z)\begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \\
\left[T_k(z)\begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2
\end{cases} \\
&= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}}
\begin{cases}
[R(z)P^\infty(z)]_{11} & z \in \Omega_\infty \cup (\Omega_0 \setminus \mathcal{U}_1) \\
\left[R(z)P^\infty(z)\begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \setminus \mathcal{U}_1 \\
\left[R(z)P^\infty(z)\begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2 \setminus \mathcal{U}_1 \\
\left[R(z)P^0(z)\right]_{11} & z \in \Omega_0 \cap \mathcal{U}_1 \\
\left[R(z)P^0(z)\begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \cap \mathcal{U}_1 \\
\left[R(z)P^0(z)\begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2 \cap \mathcal{U}_1 \,.
\end{cases}
\end{aligned}
$$

From the above relation we obtain the expansions in the following regions.

The exterior region Ω_∞.
In this region, from (2.50) and (2.51), for any integer $M \geq 2$ we have

$$
\begin{aligned}
\pi_k(z) &= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\gamma}\left(1+\mathcal{O}\left(\frac{1}{k^{M+\gamma}}\right)\right) \\
&= z^k\left(1-\frac{1}{z}\right)^{\gamma}\left(1+\mathcal{O}\left(\frac{1}{k^{M+\gamma}}\right)\right), \qquad (2.52)
\end{aligned}
$$

on any compact subset of Ω_∞. Therefore there are no zeroes accumulating in this region.

The interior region $\Omega_0 \setminus \mathcal{U}_1$.

$$\pi_k(z) = e^{kg(z)}\left(\frac{1}{k^{1+\gamma}}\frac{1}{(z-1)\Gamma(-\gamma)}\left(1-\frac{1}{z_0}\right)^{-\gamma-1} + \mathcal{O}\left(\frac{1}{k^{2+\gamma}}\right)\right)$$

The leading term of the above expansion is of order $O(k^{-1-\gamma})$, so there are no zeroes in this region.

The interesting region $\Omega_1 \setminus \mathcal{U}_1$.

$$\pi_k(z) = e^{kg(z)}\left(1-\frac{1}{z}\right)^{\gamma}\left[e^{k\phi(z)} - \frac{1}{k^{1+\gamma}}\frac{\left(1-\frac{1}{z}\right)^{-\gamma}}{(z-1)\Gamma(-\gamma)}\left(1-\frac{1}{z_0}\right)^{-\gamma-1} + \mathcal{O}\left(\frac{1}{k^{2+\gamma}}\right)\right] \tag{2.53}$$

where $e^{k\phi(z)}$ is uniformly bounded on $\Omega_1 \subset \{\mathrm{Re}(\phi) \le 0\}$.

The other interesting region $z \in \Omega_2 \setminus \mathcal{U}_1$:

$$\pi_k(z) = e^{kg(z)}\left(1-\frac{1}{z}\right)^{\gamma}\left[1 - \frac{e^{-k\phi(z)}}{k^{\gamma+1}}\left[\frac{\left(1-\frac{1}{z}\right)^{-\gamma}}{(z-1)\Gamma(-\gamma)}\left(1-\frac{1}{z_0}\right)^{-\gamma-1} + \mathcal{O}\left(\frac{1}{k}\right)\right]\right] \tag{2.54}$$

where $e^{-k\phi(z)}$ is uniformly bounded on $\Omega_2 \subset \{\mathrm{Re}(\phi) \ge 0\}$.

The region $\mathcal{U}_1$: for $z \in \mathcal{U}_1 \cap \mathrm{Ext}(\Gamma)$

$$\pi_k(z) = e^{kg(z)}\frac{(z-1)^{\gamma}}{(zw(z))^{\gamma}}\left[(w(z))^{\gamma} - \frac{e^{-k\phi(z)}}{k^{\gamma}}\left(\tilde{\Psi}_{12}(kw(z)) - \frac{w(z)^{\gamma}}{k}\left(\frac{\left(1-\frac{1}{z_0}\right)^{-\gamma-1}}{(z-1)\Gamma(-\gamma)} + v_R^{(1)}(z) + \mathcal{O}\left(\frac{1}{k}\right)\right)\right)\right]$$

for $z \in \mathcal{U}_1 \cap \mathrm{Int}(\Gamma)$

$$\pi_k(z) = e^{kg(z)}\frac{(z-1)^{\gamma}}{(zw(z))^{\gamma}}\left[e^{k\phi(z)}(w(z))^{\gamma} - \frac{\tilde{\Psi}_{12}(kw(z))}{k^{\gamma}} - \frac{w(z)^{\gamma}}{k^{1+\gamma}}\left(\frac{\left(1-\frac{1}{z_0}\right)^{-\gamma-1}}{(z-1)\Gamma(-\gamma)} + v_R^{(1)}(z) + \mathcal{O}\left(\frac{1}{k}\right)\right)\right]$$

where $\tilde{\Psi}_{12}$ is the entry 12 of the matrix $\tilde{\Psi}$ defined in (2.38) and $v_R^{(1)}(z)$ has been defined in (2.47). Here $\mathrm{Int}(\Gamma)$ and $\mathrm{Ext}(\Gamma)$ is the interior and exterior of Γ respectively.

Proposition 2.3. *The support of the counting measure of the zeroes of the polynomials $\pi_k(z)$ outside an arbitrary small disk $\mathcal{U}_1$ surrounding the point $z = 1$ tends uniformly to the curve Γ defined in* (2.98)*. The zeroes are within a distance $\mathcal{O}\left(\frac{1}{k}\right)$ from the curve defined by*

$$\mathrm{Re}\,\phi(z) = -(1+\gamma)\frac{\log(k)}{k} + \frac{1}{k}\log\left(\frac{1}{|\Gamma(-\gamma)|}\frac{|z|^{\gamma}}{|z-1|^{\gamma+1}}\left|1-\frac{1}{z_0}\right|^{-\gamma-1}\right) \tag{2.55}$$

where the function $\phi(z)$ has been defined in (2.31)*. Such curves tends to Γ at a rate $\mathcal{O}\left(\frac{\log k}{k}\right)$. The normalised counting measure of the zeroes of $\pi_k(z)$ converges to the probability measure ν defined in (2.26).*

Proof. Observing the asymptotic expansion (2.52) of $\pi_k(z)$ in $\Omega_\infty \setminus \mathcal{U}_1$ it is clear that $\pi_k(z)$ does not have any zeroes in that region, since $z = 0$ and $z = 1$ are not points in $\Omega_\infty \setminus \mathcal{U}_1$. The same reasoning applies to the region $\Omega_0 \setminus \mathcal{U}_1$ where there are no zeroes of $\pi_k(z)$.

From the relations (2.53) and (2.54) one has that in $\Omega_1 \cup \Omega_2$ using the explicit expression of $g(z)$ defined in (2.27)

$$\pi_k(z) = z^k \left(1 - \frac{1}{z}\right)^\gamma \left[1 - \frac{e^{-k\phi(z)}}{k^{\gamma+1}} \left[\frac{\left(1-\frac{1}{z}\right)^{-\gamma}}{(z-1)\Gamma(-\gamma)} \left(1 - \frac{1}{z_0}\right)^{-\gamma-1} + \mathcal{O}\left(\frac{1}{k}\right)\right]\right].$$

The zeroes of $\pi_k(z)$ may lie where the following term

$$1 - \frac{e^{-k\phi(z)}}{k^{\gamma+1}} \frac{\left(1-\frac{1}{z}\right)^{-\gamma}}{(z-1)\Gamma(-\gamma)} \left(1 - \frac{1}{z_0}\right)^{-\gamma-1}, \quad z \in \Omega_1 \cup \Omega_2,$$

is equal to zero. Since $\Omega_2 \subset \{\mathrm{Re}(\phi) \geq 0\}$ and $\Omega_1 \subset \{\mathrm{Re}(\phi) \leq 0\}$, it follows that the zeroes of $\pi_k(z)$ may lie only in the region Ω_1 and such that $\mathrm{Re}\,\phi(z) = \mathcal{O}\left(\frac{\log k}{k}\right)$. Namely the zeroes of the polynomials $\pi_k(z)$ lie on the curve given by (2.55) with an error of order $\mathcal{O}\left(\frac{1}{k}\right)$. Such curves converges to the curve Γ defined (2.98) at a rate $\mathcal{O}\left(\frac{\log k}{k}\right)$ (see figure 2.14).

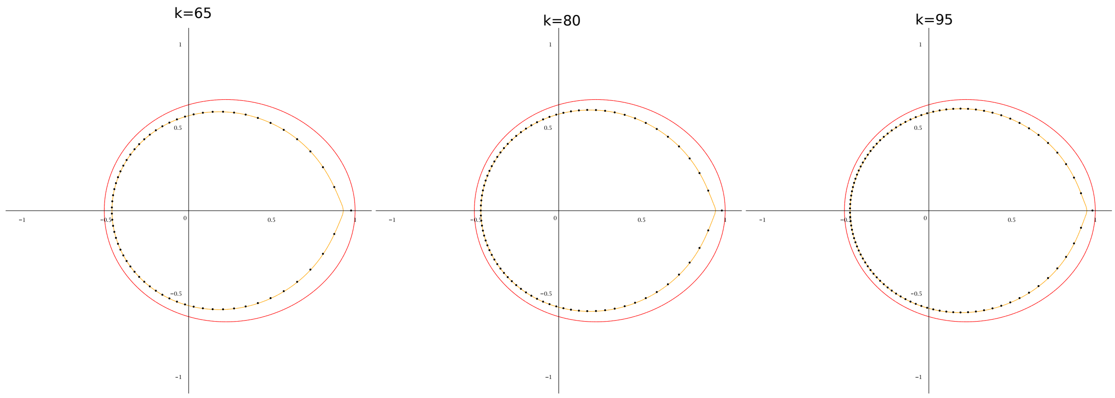


Figure 2.7: The zeroes of $\pi_k(z)$ for $s = 3$, $l = 0$, $t < t_c$, and $k = 65, 80, 95$. The red contour is Γ while the yellow contour is the curve (2.55).

In order to show that the counting measure of the zeroes converges weakly to the measure ν defined in (2.26) we observe that due to the strong asymptotic of $\pi_k(z)$ we have

$$\lim_{k\to\infty} \frac{1}{k} \log(\pi_k(z)) = \int_\Gamma \log(z - \xi) d\nu(\xi),$$

uniformly on compact subsets of the exterior of Γ. Then it follows that the measure $d\nu$ in (2.26) is the asymptotic distribution of the zeroes of $\pi_k(z)$ (see [29] Theorem 2.3). □

2.4 Asymptotic analysis: post-critical case

In this section we assume that

$$0 < z_0 < 1\ .$$

Also in this case one has to chose the right contour. We need to chose a contour surrounding the cut $[0,1]$ that is in the same homotopy class of the contour Σ. Since the point z_0 is lying in $(0,1)$, the family of curves Γ_r , $0 < r \leq z_0$ defined in (2.30) are loops encircling $z = 0$ and crossing the real line in $z = r$. The regions where $\operatorname{Re}\phi_r(z) < 0$ with $\phi_r(z)$ defined in (2.28), are depicted in the figures below in red for two values of the parameter r.

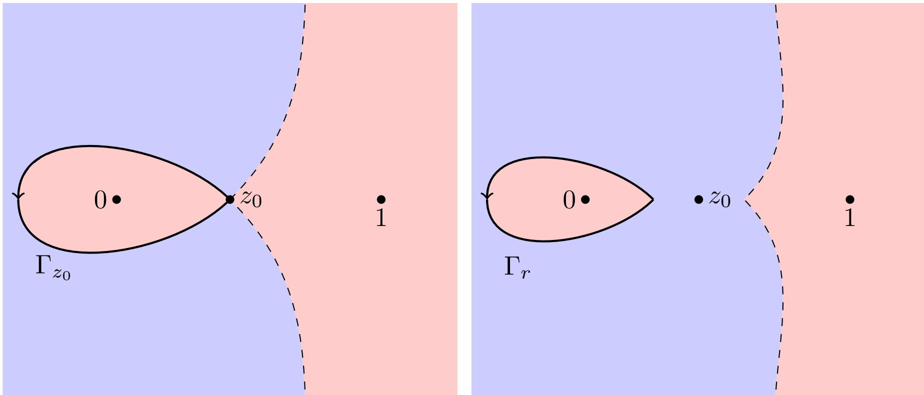


Figure 2.8: The contour Γ_r for $r = z_0$ on the left and for $0 < r < z_0$ on the right. The region where $\operatorname{Re}\phi_r(z) < 0$, with $\phi_r(z)$ defined in (2.28), is red.

In order to perform the asymptotic analysis of the Riemann-Hilbert problem (2.15), (2.16) and (2.17), we need to deform Σ to a homotopic contour $\Gamma_r \cup \Gamma_o$ in such a way that the $\operatorname{Re}\phi_r(z)$ is negative on Γ_o.

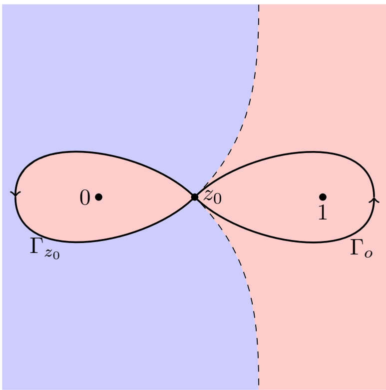


Figure 2.9: The contour $\Gamma_{z_0} \cup \Gamma_o$. The region where $\operatorname{Re}\phi_{z_0}(z) < 0$ is plotted in red.

It is clear from Figure 2.8 that the only possibility is to deform the contour Σ to the contour $\Gamma_{z_0} \cup \Gamma_o$ with Γ_o as depicted in the Figure 2.9. For simplifying the notation we define the following.

Definition 2.1. *Let*

$$\phi(z) := \phi_{z_0}(z) = \log\left(\frac{z}{z_0}\right) - \frac{z}{z_0} + 1 \qquad z \in \mathbb{C} \setminus (0, +\infty) \ , \tag{2.56}$$

with

$$\phi_+(z) - \phi_-(z) = -2\pi i \quad z \in (0, +\infty) \ .$$

and

$$\Gamma := \Gamma_{z_0}$$

with $\phi_{z_0}(z)$ *defined in* (2.28) *and* Γ_{z_0} *defined in* (2.30).

According to (2.29) on the contour Γ we have

$$\begin{aligned} g_+(z) + g_-(z) &= V(z) + \ell \\ g_+(z) - g_-(z) &= -\phi(z) \ , \end{aligned}$$

where

$$\ell = \log(z_0) - 1 \ .$$

2.4.1 First transformation $\tilde{Y} \mapsto S$

Since $\Gamma \cup \Gamma_o$ is homotopically equivalent to Σ in $\mathbb{C} \setminus [0,1]$ when $z_0 < 1$, we can deform the contour Σ appearing in the Riemann–Hilbert problem (2.15)-(2.17) for the matrix $Y(z)$ to $\Gamma \cup \Gamma_o$. Define the modified matrix

$$S(z) = e^{-k(\ell/2)\sigma_3}\tilde{Y}(z)e^{-kg(z)\sigma_3}e^{k(\ell/2)\sigma_3} \qquad z \in \mathbb{C} \setminus (\Gamma \cup \Gamma_o \cup [0,1]),$$

where $\tilde{Y}(z)$ has been defined in (2.18). Then the matrix $S(z)$ is the unique solution of the following Riemann-Hilbert problem with *standard large z behaviour at $z = \infty$*:

- $S(z)$ is analytic for $z \in \mathbb{C} \setminus (\Gamma \cup \Gamma_o \cup [0,1])$,.
- The jump on Γ:
$$S_+(z) = S_-(z) \begin{pmatrix} e^{k\phi(z)} & 1 \\ 0 & e^{-k\phi(z)} \end{pmatrix} \ ,$$
with $\phi(z)$ as in (2.56).
- The jump on Γ_o:
$$S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{k\phi(z)} \\ 0 & 1 \end{pmatrix} \ .$$
- The jump on $(0,1)$:
$$S_+(z) = S_-(z)e^{-\gamma\pi i\sigma_3} \ .$$
- Endpoint behaviour:
$$S(z)z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0 \ , \qquad S(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1 \ .$$
- Large z boundary behaviour:
$$S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \ , \quad z \to \infty \ .$$

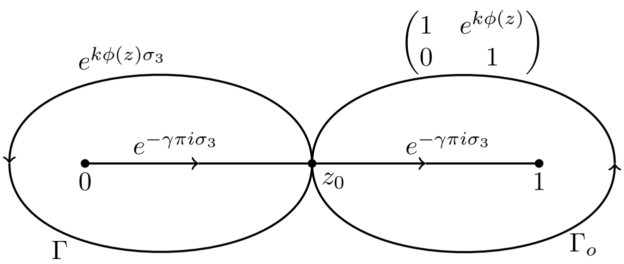


Figure 2.10: The jump matrices for $S(z)$. The function $\phi(z)$ is defined in (2.56).

The orthogonal polynomials $\pi_k(z)$ are recovered from the matrix $S(z)$ using the relation

$$\pi_k(z) = e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} S_{11}(z).$$

2.4.2 The second transformation $S \mapsto T$: opening of the lenses

Consider two extra loops Γ_i and Γ_e as shown in Figure 2.11. These define new domains Ω_0, Ω_1, Ω_2 and Ω_∞. Define

$$T(z) = \begin{cases} S(z) & z \in \Omega_\infty \cup \Omega_0 \cup \Omega_3 \\ S(z) \begin{pmatrix} 1 & 0 \\ -e^{k\phi(z)} & 1 \end{pmatrix} & z \in \Omega_1 \\ S(z) \begin{pmatrix} 1 & 0 \\ e^{-k\phi(z)} & 1 \end{pmatrix} & z \in \Omega_2 \; . \end{cases}$$

Then this matrix-valued function has the following jump discontinuities:

$$T_+(z) = T_-(z) v_T(z), \quad z \in \Sigma_T$$

$$v_T(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in \Gamma \\ \begin{pmatrix} 1 & 0 \\ e^{-k\phi(z)} & 1 \end{pmatrix} & z \in \Gamma_e \\ \begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix} & z \in \Gamma_i \\ \begin{pmatrix} 1 & e^{k\phi(z)} \\ 0 & 1 \end{pmatrix} & z \in \Gamma_o \\ e^{-\gamma\pi i\sigma_3} & z \in (0,1)\,, \end{cases}$$

where Σ_T is the contour defined in the Figure 2.11.

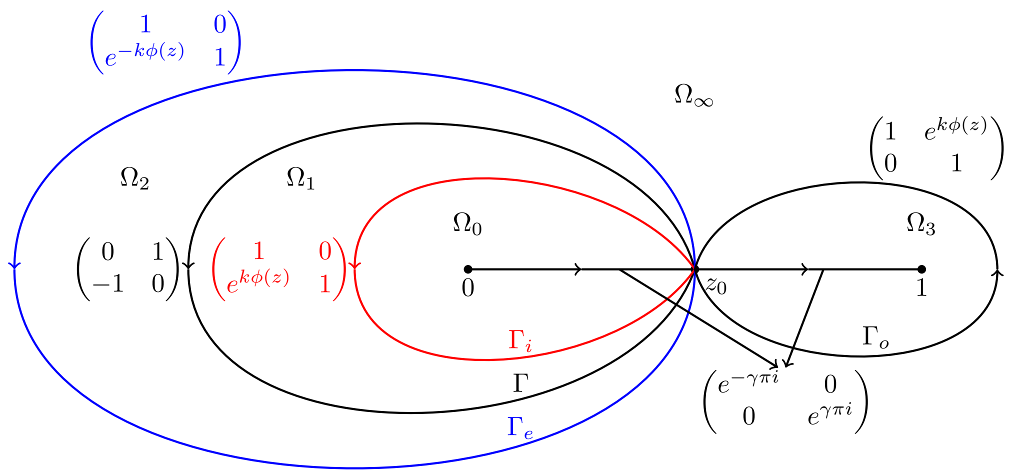


Figure 2.11: The jump matrices for $T(z)$ and the contour $\Sigma_T = \Gamma \cup \Gamma_i \cup \Gamma_e \cup \Gamma_o \cup (0,1)$.

Endpoint behaviour:

$$T(z) z^{-\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 0\,, \qquad T(z)(z-1)^{\frac{\gamma}{2}\sigma_3} = \mathcal{O}(1) \quad z \to 1\,.$$

Large z boundary behaviour:

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)\,, \quad z \to \infty\,.$$

Note that $e^{k\phi(z)}$ is analytic in $\mathbb{C} \setminus \{0\}$. The important feature of this RHP is that the jumps are either constant or exponentially decaying in k (though not uniformly).

The polynomials $\pi_k(z)$ can be expressed in terms of T in the following way

$$\begin{aligned}
\pi_k(z) &= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}} S_{11}(z) = \\
&= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}} \begin{cases} T_{11}(z) & z \in \Omega_\infty \cup \Omega_0 \cup \Omega_3 \\ \left[T(z)\begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \\ \left[T(z)\begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2 \end{cases} \\
&= e^{kg(z)}\left(1-\frac{1}{z}\right)^{\frac{\gamma}{2}} \begin{cases} T_{11}(z) & z \in \Omega_\infty \cup \Omega_0 \cup \Omega_3 \\ T_{11}(z) + e^{k\phi(z)} T_{12}(z) & z \in \Omega_1 \\ T_{11}(z) - e^{-k\phi(z)} T_{12}(z) & z \in \Omega_2\,. \end{cases}
\end{aligned} \tag{2.57}$$

Proposition 2.4. *There exists a constant $c_0 > 0$ so that*

$$v_T(z) = v^\infty(z)(I + e^{-c_0 k}) \quad \textit{as } k \to \infty$$

uniformly for $z \in \Sigma_T \backslash \mathcal{U}_{z_0}$, where $\mathcal{U}_{z_0}$ is a small neighbourhood of z_0 and

$$v^\infty(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \textit{as } z \in \Gamma_k, \\ \\ e^{-\pi i \gamma \sigma_3} & \textit{as } z \in (0,1). \end{cases} \tag{2.58}$$

2.4.3 The outer parametrix

Ignoring the exponentially small jumps and a small neighborhood $\mathcal{U}_{z_0}$ of z_0 where the uniform exponential decay does not remain valid, we are led to the following RH problem for $P^{(\infty)}$,

(a) $P^{(\infty)} : \mathbb{C} \setminus \{[0,1] \cup \Gamma\} \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $P^{(\infty)}$ satisfies the following jump conditions on Γ and $(0,1)$:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{as } z \in \Gamma,$$
$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{-\pi i \gamma \sigma_3} \qquad \text{as } z \in (0,1).$$

(c) $P^{(\infty)}(z)$ has the following behavior as $z \to \infty$,

$$P^{\infty}(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \ , \quad z \to \infty$$

The above Riemann-Hilbert problem can be solved explicitly. Indeed let us consider the matrix valued function

$$\chi(z) := \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & z \in \mathrm{int}(\Gamma) \\ I & z \in \mathrm{ext}(\Gamma) \ . \end{cases}$$

Define

$$\tilde{P}^{\infty}(z) := P^{\infty}(z)\chi^{-1}(z) \ .$$

The matrix $\tilde{P}^{\infty}(z)$ has no jump on Γ and it satisfies the following RHP:

- $\tilde{P}^{\infty}$ is holomorphic in $\mathbb{C} \setminus (0,1)$
- jump across $(0,1)$:

$$\tilde{P}_+^{\infty}(z) = \begin{cases} \tilde{P}_-^{\infty}(z) e^{\gamma \pi i \sigma_3} & z \in (0, z_0) \\ \tilde{P}_-^{\infty}(z) e^{-\gamma \pi i \sigma_3} & z \in (z_0, 1) \end{cases}$$

- large z boundary behaviour:

$$\tilde{P}^{\infty}(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \ , \quad z \to \infty \ .$$

A particular solution is given by

$$\tilde{P}^{\infty}(z) = \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}\sigma_3}$$

which leads to the particular solution

$$
P^{\infty}(z)=\left[\frac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}\sigma_3}\chi(z)=\begin{cases}\left[\dfrac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}\sigma_3}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} & z\in\mathrm{int}(\Gamma)\\ \left[\dfrac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}\sigma_3} & z\in\mathrm{ext}(\Gamma)\ .\end{cases} \tag{2.59}
$$

2.4.4 The local parametrix at $z = z_0$

The aim of this section is to construct a local parametrix $P^0(z)$ in a small neighborhood $\mathcal{U}_{z_0}$ of z_0 having the same jump property as T for z near z_0 and matching the outer parametric $P^\infty(z)$ in the limit $k \to \infty$ and $z \in \partial\mathcal{U}_{z_0}$.

RH problem for $P^0(z)$

(a) $P^0(z)$ is analytic for $z \in \overline{\mathcal{U}}_{z_0} \setminus \Sigma_T$,

(b) $P^0_+(z) = P^0_-(z)v_T(z)$ for $z \in \mathcal{U}_{z_0} \cap \Sigma_T$,

(c) for $z \in \partial\mathcal{U}_{z_0}$, we have

$$
P^0(z) = P^\infty(z)(I + o(1)) \qquad \text{as } k \to \infty \text{ and } z \in \partial\mathcal{U}_{z_0}. \tag{2.60}
$$

In order to build such local parametrix near the point $z = z_0$ we first construct a new matrix function $V(z)$ from $P^0(z)$. Let us first define

$$
\Delta(z) = \begin{cases} I & \mathrm{Im}(z) < 0 \\ e^{-\gamma\pi i\sigma_3} & \mathrm{Im}(z) > 0, \end{cases} \tag{2.61}
$$

and the matrix $\mathcal{Q}$ as follows

$$
\mathcal{Q}(z) = \begin{cases} \begin{pmatrix} 1 & -e^{k\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z \in \Omega_3 \cap \mathcal{U}_{z_0} \\ \begin{pmatrix} 1 & 0 \\ -e^{k\phi(z)} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } z \in \Omega_0 \cap \mathcal{U}_{z_0} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } z \in (\Omega_1 \backslash \Omega_0) \cap \mathcal{U}_{z_0} \\ I & \text{elsewhere.} \end{cases} \tag{2.62}
$$

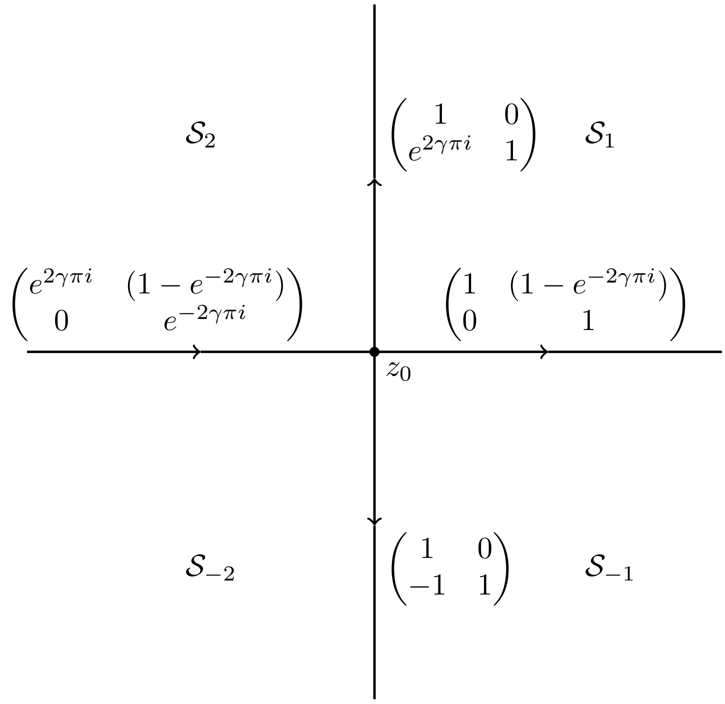


Figure 2.12: The jumps for the matrix V

Then the matrix $V(z)$ is defined from $P^0(z)$ by the relation

$$V(z) = P^0(z)\mathcal{Q}(z)e^{k\phi(z)\sigma_3/2}\Delta(z)^{-1}.$$

The matrix $V(z)$ satisfies the jump relations specified in Figure 2.12 in a neighbourhood of z_0

Model problem

Consider the model problem for the 2×2 matrix function $\Psi(\xi)$ analytic in $\mathbb{C}\backslash\{\mathbb{R} \cup i\mathbb{R}\}$ with boundary behaviour

$$\Psi_+(\xi) = \Psi_-(\xi)v_\Psi(\xi) \tag{2.63}$$

$$\Psi(\xi) = \left(I + \frac{\Psi_1}{\xi} + \frac{\Psi_2}{\xi^2} + \frac{\Psi_3}{\xi^3} + \mathcal{O}\left(\frac{1}{\xi^4}\right)\right)\left[e^{-\frac{\xi^2}{2}}\xi^\gamma\right]^{\sigma_3} \qquad \xi \to \infty\,, \tag{2.64}$$

with Ψ_1 Ψ_2 and Ψ_3 constant matrices (independent from ξ) and where the matrix $v_\Psi(\xi)$ is defined as

$$v_\Psi(\xi) = \begin{cases} \begin{pmatrix} 1 & 1-e^{-2\gamma\pi i} \\ 0 & 1 \end{pmatrix} & \xi \in \mathbb{R}^+ \\ \begin{pmatrix} 1 & 0 \\ e^{2\gamma\pi i} & 1 \end{pmatrix} & \xi \in i\mathbb{R}^+ \\ \begin{pmatrix} e^{2\gamma\pi i} & 1-e^{-2\gamma\pi i} \\ 0 & e^{-2\gamma\pi i} \end{pmatrix} & \xi \in \mathbb{R}^- \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \xi \in -i\mathbb{R}^+ \end{cases}$$

The solution of the Rieman-Hilbert problem (2.63) and (2.64) is obtained in the following way, [9, 22, 30]. Let us introduce the parabolic cylinder equation (see Chapter 19 of Abramowitz and Stegun, Handbook of Mathematical Functions, Dover, 1965)

$$\frac{d^2}{d\xi^2} f - \left(\frac{1}{4}\xi^2 + a\right) f = 0. \tag{2.65}$$

Such equation has a non zero solution $\mathcal{U}(a,\xi)$ that is, for every value of the parameter $a \in \mathbb{C}$ an entire analytic function of ξ. There are other three solutions obtained by symmetry:

$$\mathcal{U}(a,-\xi), \quad \mathcal{U}(-a,i\xi), \quad \mathcal{U}(-a,-i\xi).$$

The relations among the above four solutions are

$$\mathcal{U}(-a,\pm i\xi) = \frac{\Gamma(\frac{1}{2}+a)}{\sqrt{2\pi}}\left(e^{-i\pi(a-\frac{1}{2})/2}\mathcal{U}(a,\pm\xi) + e^{i\pi(a-\frac{1}{2})/2}\mathcal{U}(a,\mp\xi)\right) \tag{2.66}$$

$$\mathcal{U}(a,\pm\xi) = \frac{\Gamma(\frac{1}{2}-a)}{\sqrt{2\pi}}\left(e^{-i\pi(a+\frac{1}{2})/2}\mathcal{U}(-a,\pm i\xi) + e^{i\pi(a+\frac{1}{2})/2}\mathcal{U}(-a,\mp i\xi)\right) \tag{2.67}$$

$$\frac{d}{d\xi}\mathcal{U}(a,\xi) + \frac{\xi}{2}\mathcal{U}(a,\xi) + (a+\frac{1}{2})\mathcal{U}(a+1,\xi) = 0. \tag{2.68}$$

Furthermore the following asymptotic condition holds

$$\mathcal{U}(a,\xi) = \xi^{-a-\frac{1}{2}}e^{-\frac{\xi^2}{4}}\left(1 - \frac{\frac{3}{4}+a^2+2a}{2\xi^2}\right), \quad \xi\to\infty, |\arg(\xi)| < \frac{\pi}{2}. \tag{2.69}$$

Using (2.66)-(2.68), the solution of the Rieman-Hilbert problem (2.63) and (2.64) takes the form

$$\Psi(\xi) = \begin{cases} \begin{pmatrix} \mathcal{U}(-\gamma-\frac{1}{2},\sqrt{2}\xi) & \mp\frac{i\gamma e^{\mp\frac{i\pi\gamma}{2}}}{\sqrt{2}\beta_{21}}\mathcal{U}(\gamma+\frac{1}{2},\mp i\sqrt{2}\xi) \\ \frac{\gamma}{\sqrt{2}\beta_{12}}\mathcal{U}(-\gamma+\frac{1}{2},\sqrt{2}\xi) & e^{\mp\frac{i\pi\gamma}{2}}\mathcal{U}(\gamma-\frac{1}{2},\mp i\sqrt{2}\xi) \end{pmatrix} 2^{-\frac{\gamma}{2}\sigma_3}, & \text{for } \xi\in\mathcal{S}_{\pm1} \\ \begin{pmatrix} e^{\pm i\pi\gamma}\mathcal{U}(-\gamma-\frac{1}{2},-\sqrt{2}\xi) & \mp\frac{i\gamma e^{\mp\frac{i\pi\gamma}{2}}}{\sqrt{2}\beta_{21}}\mathcal{U}(\gamma+\frac{1}{2},\mp i\sqrt{2}\xi) \\ -\frac{\gamma e^{\pm i\pi\gamma}}{\sqrt{2}\beta_{12}}\mathcal{U}(-\gamma+\frac{1}{2},-\sqrt{2}\xi). & e^{\mp\frac{i\pi\gamma}{2}}\mathcal{U}(\gamma-\frac{1}{2},\mp i\sqrt{2}\xi) \end{pmatrix} 2^{-\frac{\gamma}{2}\sigma_3} & \text{for } \xi\in\mathcal{S}_{\pm2} \end{cases} \tag{2.70}$$

where

$$\beta_{12} = -e^{-i\pi\gamma}\frac{\sqrt{\pi}\gamma}{\Gamma(1-\gamma)2^\gamma} = \frac{\gamma}{2\beta_{21}}. \tag{2.71}$$

Furthermore, from (2.69) one obtains the extra terms of the asymptotic expansion of $\Psi(\xi)$ as $\xi\to\infty$

$$\Psi(\xi) = \left(I + \frac{\gamma}{2}\begin{bmatrix} 0 & \frac{1}{\beta_{21}} \\ \frac{1}{\beta_{12}} & 0 \end{bmatrix}\frac{1}{\xi} + \begin{bmatrix} -\gamma(\gamma-1) & 0 \\ 0 & \gamma(\gamma+1) \end{bmatrix}\frac{1}{4\xi^2} \right.$$
$$\left. + \frac{\gamma}{8}\begin{bmatrix} 0 & \frac{(\gamma+1)(\gamma+2)}{\beta_{21}} \\ -\frac{(\gamma-1)(\gamma-2)}{\beta_{12}} & 0 \end{bmatrix}\frac{1}{\xi^3}\right)\xi^{\gamma\sigma_3}e^{-\frac{\xi^2}{2}\sigma_3}, \tag{2.72}$$

namely

$$\begin{aligned}\Psi_1 &= \frac{\gamma}{2}\begin{bmatrix} 0 & \frac{1}{\beta_{21}} \\ \frac{1}{\beta_{12}} & 0 \end{bmatrix}, \quad \Psi_2 = \frac{1}{4}\begin{bmatrix} -\gamma(\gamma-1) & 0 \\ 0 & \gamma(\gamma+1) \end{bmatrix}, \\ \Psi_3 &= \frac{\gamma}{8}\begin{bmatrix} 0 & \frac{(\gamma+1)(\gamma+2)}{\beta_{21}} \\ -\frac{(\gamma-1)(\gamma-2)}{\beta_{12}} & 0 \end{bmatrix}.\end{aligned} \tag{2.73}$$

Construction of the parametrix

We are now ready to specify the form of the parametric $P^0(z)$

$$P^0(z) = E(z)\Psi(\sqrt{k}w(z))\Delta(z)e^{-\frac{k}{2}\phi(z)\sigma_3}\mathcal{Q}(z)^{-1}$$

where $E(z)$ is an analytic matrix in a neighbourhood of $\mathcal{U}_{z_0}$, the matrix Ψ has been defined in (2.70) and $\Delta(z)$ and $\mathcal{Q}(z)$ have been defined in (2.61) and (2.62) respectively. The function $w(z)$ is a conformal mapping from a neighbourhood of z_0 to a neighbourhood of 0 and it is specified by

$$w^2(z) := \begin{cases} -\phi(z) - 2\pi i & z \in \mathcal{U}_{z_0} \cap \mathbb{C}_- \\ -\phi(z) & z \in \mathcal{U}_{z_0} \cap \mathbb{C}_+ \, . \end{cases}$$

We observe that

$$w(z) = \frac{1}{\sqrt{2}z_0}(z-z_0) - \frac{1}{3\sqrt{2}z_0^2}(z-z_0)^2 + \mathcal{O}\left((z-z_0)^3\right) \qquad z \to z_0 \, . \tag{2.74}$$

The matrix $E(z)$ is obtained from condition (2.60) which, when combined with (2.72) gives

$$\begin{aligned} P^\infty(z)(P^0(z))^{-1} &= P^\infty(z)\mathcal{Q}(z)e^{\frac{k}{2}\phi(z)\sigma_3}\Delta(z)^{-1}\Psi(\sqrt{k}w(z))^{-1}E(z)^{-1} \\ &= P^\infty(z)\chi(z)^{-1}\Delta(z)^{-1}(\sqrt{k}w)^{-\gamma\sigma_3}\left(I - \frac{\gamma}{2}\begin{bmatrix} 0 & \frac{1}{\beta_{21}} \\ \frac{1}{\beta_{12}} & 0 \end{bmatrix}\frac{1}{\sqrt{k}w} + O(k^{-1})\right)E^{-1}(z), \end{aligned} \tag{2.75}$$

for $k \to \infty$, $\quad z \in \partial\mathcal{U}_{z_0}$. Here we used the fact that $\mathcal{Q}(z) \to \chi(z)^{-1}$ exponentially fast as $k \to \infty$, and $z \in \partial\mathcal{U}_{z_0}$. From the above expression and (2.60), it turns out that the matrix $E(z)$ takes the form

$$E(z) = P^\infty(z)\chi(z)^{-1}\Delta(z)^{-1}(\sqrt{k}w)^{-\gamma\sigma_3} = \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}\sigma_3}\Delta(z)^{-1}(\sqrt{k}w)^{-\gamma\sigma_3} \tag{2.76}$$

We observe that the function $E(z)$ is single valued in a neighbourhood of $\mathcal{U}_{z_0}$. Indeed the boundary values of $(z-z_0)^{\gamma\sigma_3}$ and $w^{-\gamma\sigma_3}$ cancel each other. The boundary value of $(z(z-1))_+^{\gamma} = (z(z-1))_-^{\gamma}e^{2\pi i\gamma}$, so that $\Delta(z)(z(z-1))^{\frac{\gamma}{2}\sigma_3}$ remains single valued in a neighbourhood of z_0.

From the expression (2.75) and (2.76) the matching between $P^0(z)$ and $P^\infty(z)$ takes the form

$$P^\infty(z)(P^0(z))^{-1} = E(z)\left(I - \frac{\gamma}{2}\begin{bmatrix} 0 & \frac{1}{\beta_{21}} \\ \frac{1}{\beta_{12}} & 0 \end{bmatrix}\frac{1}{\sqrt{kw}} + O(k^{-1})\right)E^{-1}(z), \quad k \to \infty, \quad z \in \partial\mathcal{U}_{z_0},$$

where $\gamma \in (0,1)$. It is clear from the above expression that the entry 21 of the matrix above is not small as $k \to \infty$. For this reason we need to introduce an improvement of the parametrix.

2.4.5 Improvement of the parametrix

In order to have a uniformly small error for $k \to \infty$ we have to modify the parametrices:

$$\hat{P}^{\infty}(z) := \left(I + \frac{C}{z - z_0}\right) P^{\infty}(z) \ ,$$

where C is a nilpotent matrix to be determined and

$$\hat{P}^{0}(z) := \hat{E}(z) \begin{pmatrix} 1 & 0 \\ -\frac{\Psi_{1,21}}{\sqrt{k}w(z)} & 1 \end{pmatrix} \Psi(\sqrt{k}w(z))\Delta(z)e^{-\frac{k}{2}\phi(z)\sigma_3}\mathcal{Q}(z)^{-1}$$

where the matrix Ψ_1 has been defined in (2.64) and

$$\hat{E}(z) = \left(I + \frac{C}{z - z_0}\right) E(z).$$

With those improved definitions of the parametrices, $\hat{P}^{\infty}(z)$ and $\hat{P}^{0}(z)$ satisfy the same RH conditions as before but they might have poles at $z = z_0$. Notice that $\hat{E}_k(z)$ has also a pole at $z = z_0$. However we can choose C in such a way that $\hat{P}^0(z)$ is bounded in z_0. This is accomplished by

$$C = -E(z_0)\begin{pmatrix} 0 & 0 \\ -\frac{\Psi_{1,21}}{\sqrt{k}w'(z_0)} & 0 \end{pmatrix}\left(E'(z_0)\begin{pmatrix} 0 & 0 \\ -\frac{\Psi_{1,21}}{\sqrt{k}w'(z_0)} & 0 \end{pmatrix} + E(z_0)\begin{pmatrix} 1 & 0 \\ \frac{\Psi_{1,21}w''(z_0)}{2\sqrt{k}(w'k(z_0))^2} & 1 \end{pmatrix}\right)^{-1}$$

where $E(z_0) = \left(\frac{2z_0}{k(1-z_0)}\right)^{\frac{\gamma}{2}\sigma_3} e^{\pi i \frac{\gamma}{2}\sigma_3}$ and $E'(z_0) = \gamma\sigma_3 \frac{4z_0 - 1}{6z_0(1-z_0)} E(z_0)$. From the above relation the matrix C takes the form

$$C = \begin{pmatrix} 0 & 0 \\ ck^{\gamma-\frac{1}{2}} & 0 \end{pmatrix} \tag{2.77}$$

and

$$c = \left(\frac{1-z_0}{2z_0}\right)^{\gamma} e^{-\pi i\gamma} z_0 \Psi_{1,21} 2^{\frac{1}{2}} = -\frac{\Gamma(1-\gamma)}{\sqrt{2\pi}}\left(\frac{1-z_0}{z_0}\right)^{\gamma} z_0. \tag{2.78}$$

The improved parametrix gives the following matching between $\hat{P}^{\infty}(z)$ and $\hat{P}^{0}(z)$ as $k \to \infty$ and $z_0 \in \partial\mathcal{U}_{z_0}$

$$\begin{aligned} &\hat{P}^{\infty}(z)(\hat{P}^{0}(z))^{-1} \\ &= \hat{E}(z)\left(I - \frac{\Psi_1}{\sqrt{k}w} + \frac{\Psi_1^2 - \Psi_2}{kw^2} + \frac{\Psi_2\Psi_1 + \Psi_1\Psi_2 - \Psi_1^3 - \Psi_3}{k^{\frac{3}{2}}w^3} + O(k^{-2})\right)\begin{pmatrix} 1 & 0 \\ -\frac{\Psi_{1,21}}{\sqrt{k}w} & 1 \end{pmatrix}^{-1}\hat{E}(z)^{-1} \\ &= \hat{E}(z)\left[I - \frac{\begin{pmatrix} 0 & \Psi_{1,12} \\ 0 & 0 \end{pmatrix}}{\sqrt{k}w} - \frac{\begin{pmatrix} \Psi_{2,11} & 0 \\ 0 & \Psi_{2,22} - \frac{\gamma}{2} \end{pmatrix}}{kw^2} + \frac{\begin{pmatrix} 0 & -\Psi_{3,12} \\ \Psi_{1,21}\Psi_{2,11} - \Psi_{3,21} & 0 \end{pmatrix}}{k^{\frac{3}{2}}w^3} + O(k^{-2})\right]\hat{E}(z)^{-1} \end{aligned} \tag{2.79}$$

which shows that $\hat{P}^{\infty}(z)(\hat{P}^{0}(z))^{-1} \to I$ as $k \to \infty$ and $z_0 \in \partial\mathcal{U}_{z_0}$ for $\gamma \in [0, 1)$.

Riemann-Hilbert problem for the error matrix R

We now define the error matrix R in two regions of the plane, using our approximations to the matrix T. Set

$$R(z) = \begin{cases} T(z)\left(\hat{P}^{(0)}(z)\right)^{-1}, & z \in \mathcal{U}_{z_0}\,, \\ T(z)\left(\hat{P}^{\infty}(z)\right)^{-1}, & \text{everywhere else.} \end{cases} \tag{2.80}$$

The matrix R is piece-wise analytic in $\mathbb{C}$ with a jump across Σ_R (see Figure 2.13)

RH problem for R

(a) R is analytic in $\mathbb{C} \setminus \Sigma_R$,

(b) For $z \in \Sigma_R$, we have

$$R_+(z) = R_-(z) v_R(z), \tag{2.81}$$

with

$$v_R(z) = \hat{P}_-^{\infty}(z) v_T(z) \left(\hat{P}_+^{\infty}(z)\right)^{-1}, \qquad z \in \Sigma_R \backslash \partial\mathcal{U}_{z_0}$$
$$v_R(z) = \hat{P}^{\infty}(z)\left(\hat{P}^{(0)}(z)\right)^{-1}, \qquad z \in \partial\mathcal{U}_{z_0}$$

(c) As $z \to \infty$, we have

$$R(z) = I + O(z^{-1}).$$

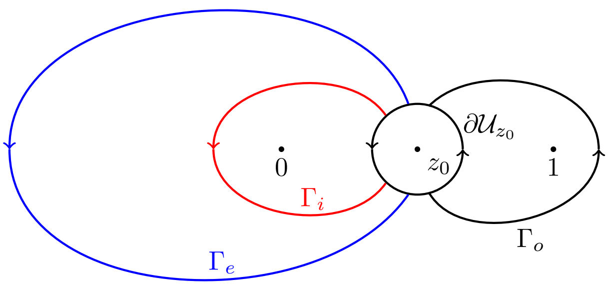


Figure 2.13: The jump contour $\Sigma_R = \Gamma_i \cup \Gamma_e \cup \Gamma_o \cup \partial\mathcal{U}_{z_0}$ for the matrix R. Here the contours Γ_i, Γ_e and Γ_o are defined only in $\mathbb{C}\backslash\mathcal{U}_{z_0}$.

The jump matrices across the contour $\Sigma_R\backslash\partial\mathcal{U}_{z_0}$ are all exponentially close to I for large k because $v_T(z)$ converges exponentially fast to $v^{\infty}(z)$ defined in (2.58) for $z \in \mathbb{C}\backslash\mathcal{U}_{z_0}$ and the product

$$\hat{P}_-^{\infty}(z) v^{\infty}(z)\left(\hat{P}_+^{\infty}(z)\right)^{-1} = \left(I + \frac{C}{z - z_0}\right) P_-^{\infty}(z) v^{\infty}(z) \left(P_+^{\infty}(z)\right)^{-1} \left(I + \frac{C}{z - z_0}\right)^{-1} = I$$

with $P^{\infty}(z)$ defined in (2.59) and C in (2.77). The only jump that is not exponentially small is the one on $\partial\mathcal{U}_{z_0}$. Indeed one has from (2.79)

$$v_R(z) = \hat{P}^{\infty}(z)(\hat{P}^0(z))^{-1} = I + \frac{\gamma e(z)^2}{2k^{\frac{1}{2}+\gamma}w(z)\beta_{21}}\begin{pmatrix} \frac{ck^{\gamma-\frac{1}{2}}}{z-z_0} & -1 \\ \frac{c^2k^{2\gamma-1}}{(z-z_0)^2} & -\frac{ck^{\gamma-\frac{1}{2}}}{z-z_0}\end{pmatrix} + \frac{\gamma(1-\gamma)}{4kw(z)^2}\begin{pmatrix} 1 & 0 \\ \frac{2ck^{\gamma-\frac{1}{2}}}{(z-z_0)} & -1\end{pmatrix}$$
$$+ \frac{1}{k^{\frac{3}{2}-\gamma}w(z)^3}\begin{pmatrix} O(k^{-\frac{1}{2}-\gamma}) & O(k^{-2\gamma}) \\ \frac{\gamma(1-\gamma)}{4\beta_{12}e(z)^2} + O(k^{-1}) & O(k^{-\frac{1}{2}-\gamma})\end{pmatrix} + O(k^{-2}, k^{\gamma-\frac{5}{2}})$$

where

$$e(z) = E_{11}(z)k^{\gamma/2} \tag{2.82}$$

and we have substituted the explicit expressions of the matrix Ψ_1, Ψ_2 and Ψ_3 as given by (2.73).

We have the following two cases depending on the value of $\gamma \in (0,1)$.

(a) $0 < \gamma < \frac{1}{2}$

$$v_R(z) = \hat{P}^{\infty}(z)(\hat{P}^0(z))^{-1} = I + \frac{v_R^1(z)}{k^{\frac{1}{2}+\gamma}} + O(k^{-1}), \tag{2.83}$$

where

$$v_R^1(z) = -\frac{\gamma e(z)^2}{2w(z)\beta_{21}}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} \tag{2.84}$$

(b) $\frac{1}{2} \leq \gamma < 1$

$$v_R(z) = \hat{P}^{\infty}(z)(\hat{P}^0(z))^{-1} = I + \frac{v_R^{(1)}(z)}{k^{\frac{3}{2}-\gamma}} + \frac{v_R^{(2)}(z)}{k} + \frac{v_R^{(3)}(z)}{k^{\frac{1}{2}+\gamma}} + O(k^{\gamma-\frac{5}{2}}). \tag{2.85}$$

where

$$\begin{aligned} v_R^{(1)}(z) &= \left(\frac{c^2}{\beta_{21}}\frac{w(z)e(z)^2}{(z-z_0)^2} + \frac{c(1-\gamma)}{2w(z)^2(z-z_0)} + \frac{(1-\gamma)}{2\beta_{12}w(z)^3e(z)^2}\right)\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} \\ v_R^{(2)}(z) &= \left(\frac{\gamma c}{2\beta_{21}}\frac{e(z)^2}{w(z)(z-z_0)} + \frac{\gamma(1-\gamma)}{4w(z)^2}\right)\sigma_3 \\ v_R^{(3)}(z) &= -\frac{\gamma e(z)^2}{2w(z)\beta_{21}}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \end{aligned} \tag{2.86}$$

By standard procedure for small norm RH problem one has a similar expansion for R in the large k limit.

Case (a): $0 < \gamma < \frac{1}{2}$

$$R(z) = I + \frac{R^{(1)}}{k^{\frac{1}{2}+\gamma}} + \mathcal{O}\left(k^{-1}\right) \tag{2.87}$$

Compatibility of (2.81), (2.83) and (2.87) and the jump condition $R_+ = R_- \hat{v}_R$ on $\partial \mathcal{U}_{z_0}$ gives the following relations.

$$R_+^{(1)}(z) = R_-^{(1)}(z) + v_R^{(1)}(z), \quad z \in \partial \mathcal{U}_{z_0}$$

with $v_R^{(1)}(z)$ defined in (2.84). In addition $R^{(1)}(z)$ is analytic in $\mathbb{C} \backslash \partial \mathcal{U}_{z_0}$ and $R^{(1)}(\infty) = 0$. The unique function that satisfies those conditions is given by

$$R^{(1)}(z) = \frac{1}{2\pi i} \oint \frac{v_R^{(1)}(\xi)}{\xi - z} d\xi = \begin{cases} \dfrac{1}{z - z_0} \dfrac{\gamma e(z_0)^2}{2w'(z_0)\beta_{21}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & z \in \mathbb{C} \setminus \mathcal{U}_{z_0} \\ \\ v_R^{(1)}(z) + \dfrac{1}{z - z_0} \dfrac{\gamma e(z_0)^2}{2w'(z_0)\beta_{21}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & z \in \mathcal{U}_{z_0}, \end{cases}$$

where the integral is taken along $\partial \mathcal{U}_{z_0}$. Using the definition of β_{21} , $w'(z)$ and $e(z)$ given in (2.71), (2.74) and (2.82) respectively one has

$$R^{(1)}(z) = \begin{cases} \dfrac{1}{z - z_0} \dfrac{\gamma z_0^2}{c} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & z \in \mathbb{C} \setminus \mathcal{U}_{z_0} \\ \\ v_R^{(1)}(z) + \dfrac{1}{z - z_0} \dfrac{\gamma z_0^2}{c} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & z \in \mathcal{U}_{z_0}, \end{cases}$$

with the constant c defined in (2.78).

Case (b): $\frac{1}{2} \leq \gamma < 1$

$$R(z) = I + \frac{R^{(1)}}{k^{\frac{3}{2} - \gamma}} + \frac{R^{(2)}}{k} + \frac{R^{(3)}}{k^{\frac{1}{2} + \gamma}} + O(k^{\gamma - \frac{5}{2}}) \tag{2.88}$$

Compatibility of (2.81), (2.85) and (2.88) and the jump condition $R_+ = R_- \hat{v}_R$ on $\partial \mathcal{U}_{z_0}$ gives the following relations.

$$R_+^{(i)}(z) = R_-^{(i)}(z) + v_R^{(i)}(z), \quad z \in \partial \mathcal{U}_{z_0}$$

with $v_R^{(i)}(z)$ defined in (2.86). In addition $R^{(i)}(z)$ is analytic in $\mathbb{C} \backslash \partial \mathcal{U}_{z_0}$ and $R^{(i)}(\infty) = 0$.

Since $v_R^{(1)}(z)$ has a third order pole at $z = z_0$, the unique function that satisfies those conditions is given by

$$R^{(1)}(z) = -\frac{\operatorname{Res}_{\lambda = z_0} v_R^{(1)}(\lambda)}{z - z_0} - \frac{\operatorname{Res}_{\lambda = z_0} (\lambda - z_0) v_R^{(1)}(\lambda)}{(z - z_0)^2} - \frac{\operatorname{Res}_{\lambda = z_0} (\lambda - z_0)^2 v_R^{(1)}(\lambda)}{(z - z_0)^3} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

for $z \in \mathbb{C} \setminus \mathcal{U}_{z_0}$.

Given the structure of the matrix $\hat{P}^0(z)$, the matrix $R^{(1)}(z)$ does not give any relevant contribution to the orthogonal polynomials $\pi_k(z)$.

Regarding $R^{(2)}(z)$ one has

$$\begin{aligned}R^{(2)}(z) &= -\frac{\mathrm{Res}_{\lambda=z_0}\, v_R^{(2)}(\lambda)}{z-z_0} - \frac{\mathrm{Res}_{\lambda=z_0}(\lambda-z_0)v_R^{(2)}(\lambda)}{(z-z_0)^2} \\ &= -\frac{1}{z-z_0}\frac{c\gamma e(z_0)^2}{2\beta_{21}w'(z_0)}\left(\frac{2e'(z_0)}{e(z_0)} - \frac{w''(z_0)}{2w'(z_0)} + \frac{1}{z-z_0}\right)\sigma_3 \\ &+ \frac{1}{z-z_0}\frac{\gamma(1-\gamma)}{4w'(z_0)^2}\left(\frac{w''(z_0)}{w'(z_0)} - \frac{1}{z-z_0}\right)\sigma_3 \quad z \in \mathbb{C}\setminus\mathcal{U}_{z_0},\end{aligned}$$

so that, using (2.71), (2.74), (2.78) and (2.82) one obtains

$$R^{(2)}(z) = \begin{cases} -\dfrac{\gamma z_0^2}{z-z_0}\left(\dfrac{\gamma(4z_0-1)}{3z_0(1-z_0)} + \dfrac{2-\gamma}{3z_0} + \dfrac{3-\gamma}{2(z-z_0)}\right)\sigma_3, & z \in \mathbb{C}\setminus\mathcal{U}_{z_0} \\ \\ v_R^{(2)}(z) - \dfrac{\gamma z_0^2}{z-z_0}\left(\dfrac{\gamma(4z_0-1)}{3z_0(1-z_0)} + \dfrac{2-\gamma}{3z_0} + \dfrac{3-\gamma}{2(z-z_0)}\right)\sigma_3, & z \in \mathcal{U}_{z_0}. \end{cases}$$

In a similar way for $R^{(3)}(z)$ we obtain

$$R^{(3)}(z) = -\frac{\mathrm{Res}_{\lambda=z_0}\, v_R^{(3)}(\lambda)}{z-z_0} = \frac{1}{z-z_0}\frac{\gamma e(z_0)^2}{2w'(z_0)\beta_{21}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad z \in \mathbb{C}\setminus\mathcal{U}_{z_0},$$

so that

$$R^{(3)}(z) = \begin{cases} \dfrac{1}{z-z_0}\dfrac{\gamma z_0^2}{c}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & z \in \mathbb{C}\setminus\mathcal{U}_{z_0} \\ \\ v_R^{(3)}(z) + \dfrac{1}{z-z_0}\dfrac{\gamma z_0^2}{c}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & z \in \mathcal{U}_{z_0}, \end{cases}$$

with the constant c defined in (2.78).

2.4.6 Asymptotics for $\pi_k(z)$ for $0 < z_0 < 1$

We can now derive the asymptotic expansion of the reduced polynomials $\pi_k(z)$ as $k \to \infty$. Using the relation (2.57) and the relation (2.80) one obtains

$$\pi_k(z) = e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} \begin{cases} \left[R(z)\hat{P}^\infty(z)\right]_{11} & z \in (\Omega_\infty \cup \Omega_0 \cup \Omega_3) \setminus \mathcal{U}_{z_0} \\ \left[R(z)\hat{P}^\infty(z) \begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \setminus \mathcal{U}_{z_0} \\ \left[R(z)\hat{P}^\infty(z) \begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2 \setminus \mathcal{U}_{z_0} \\ \left[R(z)\hat{P}^0(z)\right]_{11} & z \in (\Omega_\infty \cup \Omega_0 \cup \Omega_3) \cap \mathcal{U}_{z_0} \\ \left[R(z)\hat{P}^0(z) \begin{pmatrix} 1 & 0 \\ e^{k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_1 \cap \mathcal{U}_{z_0} \\ \left[R(z)\hat{P}^0(z) \begin{pmatrix} 1 & 0 \\ -e^{-k\phi(z)} & 1 \end{pmatrix}\right]_{11} & z \in \Omega_2 \cap \mathcal{U}_{z_0}\,. \end{cases} \tag{2.89}$$

The region $(\Omega_\infty \cup \Omega_3) \setminus \mathcal{U}_{z_0}$

$$\pi_k(z) = e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{\frac{\gamma}{2}} \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right) = e^{kg(z)} \left(\frac{z-z_0}{z}\right)^{\gamma} \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right).$$

The region $\Omega_0 \setminus \mathcal{U}_{z_0}$

$$\pi_k(z) = \frac{1}{k^{\frac{1}{2}+\gamma}} e^{kg(z)} \left(\frac{\gamma z_0^2}{c} \frac{(z-1)^\gamma}{(z-z_0)^{\gamma+1}} + \mathcal{O}\left(\frac{1}{k}\right)\right).$$

with c defined in (2.78).

The region $\Omega_1 \setminus \mathcal{U}_{z_0}$

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-z_0}{z}\right)^{\gamma} \left(e^{k\phi(z)} - \begin{cases} \dfrac{1}{z-z_0} \dfrac{\gamma z_0^2}{c} \left[\dfrac{(z-z_0)^2}{z(z-1)}\right]^{-\gamma} \dfrac{1}{k^{\frac{1}{2}+\gamma}} + \mathcal{O}\left(\dfrac{1}{k}\right)\Bigg), & 0 < \gamma < \frac{1}{2} \\ \dfrac{e^{k\phi(z)}\gamma z_0^2}{z-z_0} \dfrac{1}{k} \left(\dfrac{\gamma(4z_0-1)}{3z_0(1-z_0)} + \dfrac{2-\gamma}{3z_0} + \dfrac{3-\gamma}{2(z-z_0)}\right) + \mathcal{O}\left(\dfrac{1}{k^{\frac{5}{2}-\gamma}}\right)\Bigg), & \frac{1}{2} \le \gamma < 1 \end{cases}\right.$$

with c defined in (2.78) and where we observe that $\operatorname{Re}\phi(z) \le 0$ in Ω_1. In a similar way we can obtain the expansion in the region $\Omega_2 \setminus \mathcal{U}_{z_0}$.

The region $\Omega_2 \setminus \mathcal{U}_{z_0}$

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-z_0}{z}\right)^{\gamma} \left(1 - \frac{e^{-k\phi(z)}}{k^{\frac{1}{2}+\gamma}} \frac{\gamma z_0^2}{c} \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{-\gamma} \frac{1}{z-z_0} \right.$$

$$\left.\left.\begin{array}{ll} +\mathcal{O}\left(\frac{1}{k}\right)\right), & 0<\gamma<\frac{1}{2} \\ -\frac{\gamma z_0^2}{z-z_0}\frac{1}{k}\left(\frac{\gamma(4z_0-1)}{3z_0(1-z_0)} + \frac{2-\gamma}{3z_0} + \frac{3-\gamma}{2(z-z_0)}\right) + \mathcal{O}\left(\frac{1}{k^{\frac{5}{2}-\gamma}}\right)\right), & \frac{1}{2}\le\gamma<1 \end{array}\right\}$$

where now we observe that $\operatorname{Re}\phi(z) \geq 0$ in Ω_2.

The region $\mathcal{U}_{z_0}$

Using the relations (2.89), (2.70), (2.66) and (2.67) one obtains for $z \in \mathcal{U}_{z_0} \cap \operatorname{Ext}(\Gamma)$

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-z_0}{z}\right)^{\gamma} \frac{e^{-k\phi(z)/2}}{(\sqrt{k}w(z))^{\gamma}} \left(\mathcal{U}(-\gamma-\frac{1}{2}; \sqrt{2k}w(z)) + \mathcal{O}\left(\frac{1}{k^{\frac{1}{2}}}\right)\right),$$

where $\mathcal{U}$ is the parabolic cylinder function that solves equation (2.65). For $z \in \mathcal{U}_{z_0} \cap \operatorname{Int}(\Gamma)$ we have

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-z_0}{z}\right)^{\gamma} \frac{e^{k\phi(z)/2}}{(\sqrt{k}w(z))^{\gamma}} \left(\mathcal{U}(-\gamma-\frac{1}{2}; \sqrt{2k}w(z)) + \mathcal{O}\left(\frac{1}{k^{\frac{1}{2}}}\right)\right).$$

Proposition 2.5. *The support of the counting measure of the zeroes of the polynomials $\pi_k(z)$ for $0 < t < t_c$ outside an arbitrary small disk $\mathcal{U}_{z_0}$ surrounding the point $z = z_0$ tends uniformly to the curve Γ defined in* (2.98). *The zeroes are within a distance $\mathcal{O}\left(\frac{1}{k}\right)$ from the curve defined by*

$$\operatorname{Re}\phi(z) = -\left(\frac{1}{2}+\gamma\right)\frac{\log(k)}{k} + \frac{1}{k}\log\left(\frac{\gamma z_0^2}{c}\left|\frac{(z-z_0)^2}{z(z-1)}\right|^{-\gamma}\right) \tag{2.90}$$

where the function $\phi(z)$ has been defined in (2.31). *Such curves tends to Γ at a rate $\mathcal{O}\left(\frac{\log k}{k}\right)$. The normalised counting measure of the zeroes of $\pi_k(z)$ converges to the probability measure ν defined in (2.26).*

Proof. As in the proof of Proposition 2.3, it is clear from the above expansions of the polynomials $\pi_k(z)$ that there are no zeroes in the region $(\Omega_\infty \cup \Omega_3 \cup \Omega_0) \setminus \mathcal{U}_{z_0}$. Then we observe that the asymptotic expansion of the polynomials $\pi_k(z)$ in the regions $\Omega_1 \cup \Omega_2 \setminus \mathcal{U}_{z_0}$ takes the form

$$\pi_k(z) = z^k \left(\frac{z-z_0}{z}\right)^{\gamma} \left(1 - \frac{e^{-k\phi(z)}}{k^{\frac{1}{2}+\gamma}} \frac{\gamma z_0^2}{c} \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{-\gamma} \frac{1}{z-z_0} \right.$$

$$\left.\left.\begin{array}{ll} +\mathcal{O}\left(\frac{1}{k}\right)\right), & 0<\gamma<\frac{1}{2} \\ -\frac{\gamma z_0^2}{z-z_0}\frac{1}{k}\left(\frac{\gamma(4z_0-1)}{3z_0(1-z_0)} + \frac{2-\gamma}{3z_0} + \frac{3-\gamma}{2(z-z_0)}\right) + \mathcal{O}\left(\frac{1}{k^{\frac{5}{2}-\gamma}}\right)\right), & \frac{1}{2}\le\gamma<1 \end{array}\right\}$$

so that we conclude that the zeroes of $\pi_k(z)$ occur in the region where

$$1 - \frac{e^{-k\phi(z)}}{k^{\frac{1}{2}+\gamma}} \frac{\gamma z_0^2}{c} \left[\frac{(z-z_0)^2}{z(z-1)}\right]^{-\gamma} \frac{1}{z-z_0}$$

is equal to zero.

Since $\Omega_2 \subset \{\mathrm{Re}(\phi) \geq 0\}$ and $\Omega_1 \subset \{\mathrm{Re}(\phi) \leq 0\}$, it follows that the zeroes of $\pi_k(z)$ may lie only in the region Ω_1 and such that $\mathrm{Re}\,\phi(z) = \mathcal{O}\left(\frac{\log k}{k}\right)$. Namely the zeroes of the polynomials $\pi_k(z)$ lie on the curve given by (2.55) with an error of order $\mathcal{O}\left(\frac{1}{k}\right)$. Such curves converges to the curve Γ defined (2.98) at a rate $\mathcal{O}\left(\frac{\log k}{k}\right)$ (see figure 2.14).

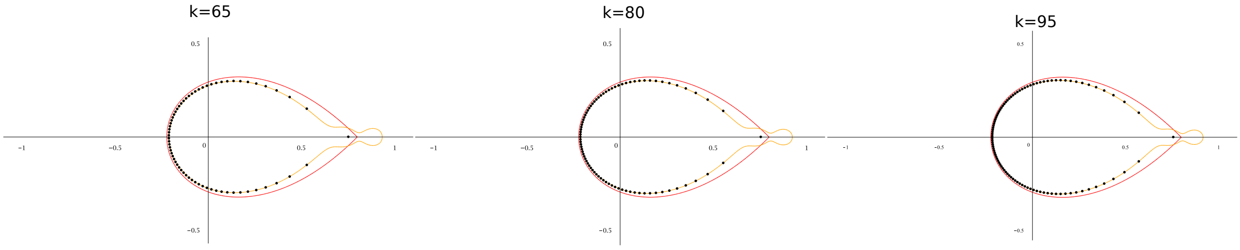


Figure 2.14: The zeroes of $\pi_k(z)$ for $s = 3$, $l = 0$, $t > t_c$ and $k = 65, 80, 95$. The red contour is Γ while the yellow contour is the curve (2.90).

The proof of the remaining points of Proposition 2.5 follows the lines of the proof of Proposition 2.3. □

2.5 Asymptotics for $p_n(\lambda)$

In order to obtain the asymptotic expansion of the polynomials $p_n(\lambda)$ for $n \to \infty$, and $NT = n - l$ we use the substitution

$$p_n(\lambda) = (-t)^k \lambda^l \pi_k\left(1 - \frac{\lambda^s}{t}\right), \quad n = ks + l, \;\; \gamma = \frac{s-1-l}{s}.$$

and use the asymptotic expansion obtained for $\pi_k(z)$.

2.5.1 Pre-critical case

Theorem 2.2. *For $0 < t < t_c$ the polynomial $p_n(\lambda)$ with $n = ks + l$, $l = 0, \ldots, s-2$, or $\gamma \in (0,1)$, have the following asymoptotic behaviour for when $n, N \to \infty$ in such a way that $NT = n - l$:*

(1) for λ in compact subsets of the exterior of $\hat{\Gamma}$ one has

$$p_n(\lambda) = \lambda^{s-1}(\lambda^s - t)^{k-\gamma}\left(1 + O\left(\frac{1}{k^{2+\gamma}}\right)\right);$$

(2) *for* λ *near* $\hat{\Gamma}$ *and away from* $\lambda = 0$,

$$p_n(\lambda) = \lambda^{s-1}(\lambda^s - t)^{k-\gamma}\left[1 + \frac{e^{-k\hat{\phi}(\lambda)}}{k^{1+\gamma}}\left(\frac{1}{\Gamma(-\gamma)}\left(1 - \frac{1}{z_0}\right)^{-1-\gamma}\frac{t}{\lambda^s}\left(1 - \frac{t}{\lambda^s}\right)^{\gamma} + O\left(\frac{1}{k}\right)\right)\right], \tag{2.91}$$

where

$$\hat{\phi}(\lambda) = \hat{\phi}_{r=z_0}(\lambda).$$

with $\hat{\phi}_r(\lambda)$ *defined in* (2.96)*;*

(3) *for* λ *in compact subsets of the interior of* $\hat{\Gamma}$ *and away from* $\lambda = 0$,

$$p_n(\lambda) = \lambda^l \frac{e^{-\frac{k\lambda^s}{tz_0}}}{k^{1+\gamma}}\left(\frac{(-t)^{k+1}}{\Gamma(-\gamma)}\frac{1}{\lambda^s}\left(1 - \frac{1}{z_0}\right)^{-1-\gamma} + O\left(\frac{1}{k}\right)\right);$$

(4) *for* λ *in a neighbourhood of* $\lambda = 0$, *we introduce the function* $\hat{w}(\lambda) = \hat{\phi}(\lambda) + 2\pi i$ *if* $\lambda^s \in \mathbb{C}_-$ *and* $\hat{w}(\lambda) = \hat{\phi}(\lambda)$ *if* $\lambda^s \in \mathbb{C}_+$. *Then*

$$p_n(\lambda) = \lambda^l(\lambda^s - t)^{k-\gamma}\left(\frac{\lambda^s}{\hat{w}(\lambda)}\right)^{\gamma}\left[(\hat{w}(\lambda))^{\gamma} - \frac{e^{-k\hat{\phi}(\lambda)}}{k^{\gamma}}\left(\tilde{\Psi}_{12}(k\hat{w}(\lambda)) + O\left(\frac{1}{k}\right)\right)\right],$$

where the the entry 12 *of the matrix* $\tilde{\Psi}$ *has been defined in* (2.38).

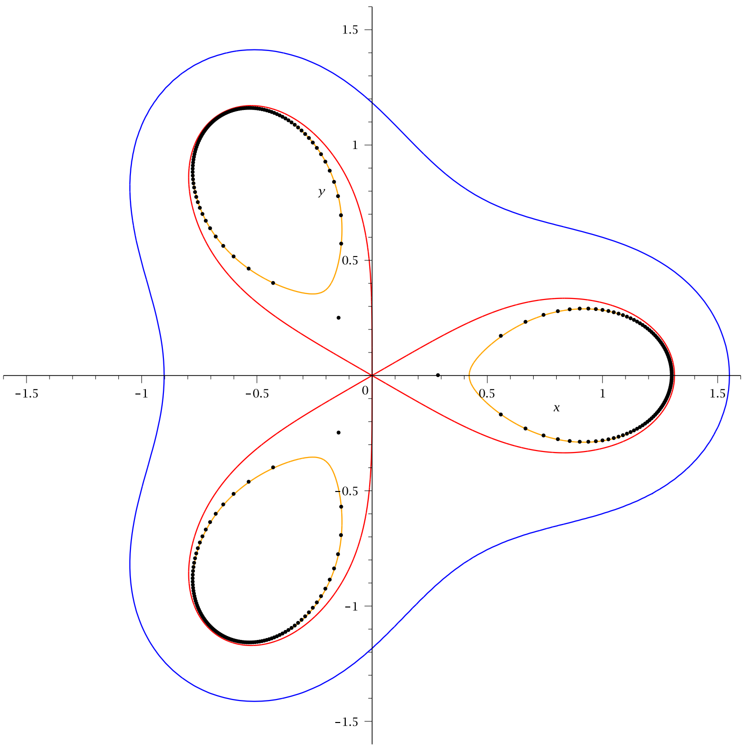


Figure 2.15: The blue contour is the boundary of the domain D defined in (2.3) and the red contour is $\hat{\Gamma}$ defined in (2.98), the yellow contour is given by (2.92) in the pre-critical case and (2.94) in the post-critical case. The dots are the zeroes of the polynomial $p_n(\lambda)$ for $s = 3$, $n = 285$, $l = 0$ and $t < t_c$.

We observe that in compact subsets of the exterior of $\hat{\Gamma}$ there are no zeroes of the polynomials $p_n(\lambda)$. The only possible zeroes are located in $\lambda = 0$ and in the region where the second term in parenthesis in the expression (2.91) is of order one. Since $\operatorname{Re}\hat{\phi}(\lambda)$ is positive inside $\hat{\Gamma}$ and negative outside $\hat{\Gamma}$ it follows that the possible zeroes of $p_n(\lambda)$ lie inside $\hat{\Gamma}$ and are determined by the condition

$$\log|\hat{\phi}(\lambda)| = -(1+\gamma)\frac{\log k}{k} + \frac{1}{k}\log\left(\frac{1}{|\Gamma(-\gamma)|}\frac{t}{|\lambda|^s}\left|1-\frac{1}{z_0}\right|^{-1-\gamma}\left|1-\frac{t}{\lambda^s}\right|^{\gamma}\right), \quad |\lambda^s - t| \le t. \tag{2.92}$$

The above expression shows that the zeroes of the polynomials $p_n(\lambda)$ are within a distance $O(1/k)$ from the level curve (2.92). Such curve converges to $\hat{\Gamma}$ defined in (2.98) at a rate $O(\log k/k)$.

2.5.2 Post-critical case

Theorem 2.3. *For $|t| > t_c$ the polynomials $p_n(\lambda)$ with $n = ks + l$, $l = 0, \ldots, s-2$, and $\gamma = \frac{s-l-1}{s} \in (0,1)$ have the following behaviour when $n, N \to \infty$ in such a way that $NT = n - l$*

(1) for λ in compact subsets of the exterior of $\hat{\Gamma}$ one has

$$p_n(\lambda) = \lambda^l(\lambda^s - t)^{k-\gamma}(\lambda^s + t(z_0-1))^{\gamma}\left(1 + O\left(\frac{1}{k}\right)\right),$$

(2) for λ in the region near $\hat{\Gamma}$ and away from the points $\lambda^s = t(1-z_0)$ one has

$$p_n(\lambda) = \lambda^l(\lambda^s - t)^{k-\gamma}(\lambda^s + t(z_0-1))^{\gamma}\left(1 - \frac{e^{-k\hat{\phi}(\lambda)}}{k^{\frac{1}{2}+\gamma}}\frac{\gamma z_0^2}{c}\frac{t((\lambda^s - t)\lambda^s)^{\gamma}}{(\lambda^s + t(z_0-1))^{2\gamma+1}} + O\left(\frac{1}{k}\right)\right) \tag{2.93}$$

where

$$\hat{\phi}(\lambda) = \hat{\phi}_{r=1}(\lambda).$$

with $\hat{\phi}_r(\lambda)$ defined in (2.96);

(3) for λ in compact subsets of the interior region of $\hat{\Gamma}$ one has

$$p_n(\lambda) = \lambda^l \frac{e^{k\frac{t-\lambda^s}{tz_0}}}{k^{\frac{1}{2}+\gamma}}\frac{t\gamma z_0^2}{c}\left(\frac{tz_0}{e}\right)^k\left(\frac{\lambda^{s\gamma}}{(\lambda^s + t(z_0-1))^{\gamma+1}} + O\left(\frac{1}{k}\right)\right);$$

(4) in the neighbourhood of each of the points that solve the equation $\lambda^s = t(1-z_0)$ one has

$$p_n(\lambda) = \lambda^l(\lambda^s - t)^{k-\gamma}\left(\frac{\lambda^s + t(z_0-1)}{\sqrt{k}\hat{w}(\lambda)}\right)^{\gamma} e^{-k\hat{\phi}(\lambda)}\left(\mathcal{U}(-\gamma - \frac{1}{2}; \sqrt{2k}\hat{w}(\lambda)) + O\left(\frac{1}{k}\right)\right);$$

where $\mathcal{U}(a;\xi)$ is the parabolic cylinder function satisfying the equation $\frac{d^2}{d\xi^2}\mathcal{U} = \left(\frac{1}{4}\xi^2 + a\right)\mathcal{U}$ and $\hat{w}^2(\lambda) = -\hat{\phi}(\lambda) - 2\pi i$ for $\lambda^s \in \mathbb{C}_-$ and $\hat{w}^2(\lambda) = -\hat{\phi}(\lambda)$ for $\lambda^s \in \mathbb{C}_+$.

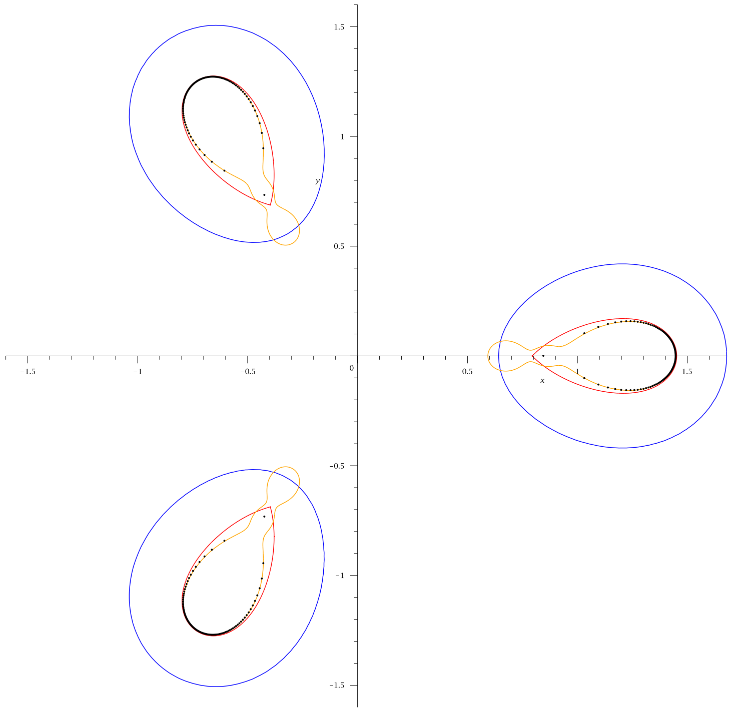


Figure 2.16: The blue contour is the boundary of the domain D defined in (2.3) and the red contour is $\hat{\Gamma}$ defined in (2.98), the yellow contour is given by (2.92) in the pre-critical case and (2.94) in the post-critical case. The dots are the zeroes of the polynomial $p_n(\lambda)$ for $s = 3$, $n = 285, l = 0$ and $t > t_c$.

We observe that in compact subsets of the exterior of $\hat{\Gamma}$ the polynomials $p_n(\lambda)$ have zero at $\lambda = 0$ with multiplicity l. The other possible zeroes are located in the region where the second term in parenthesis in the expression (2.93) is of order one. Since $\mathrm{Re}\,\hat{\phi}(\lambda)$ is positive inside $\hat{\Gamma}$, this is possible and it follows that

$$\log|\hat{\phi}(\lambda)| = -(1+\gamma)\frac{\log k}{k} + \frac{1}{k}\log\left(\frac{t\gamma z_0^2}{|c|}\frac{|(\lambda^s - t)\lambda^s|^{\gamma}}{|\lambda^s + t(z_0 - 1)|^{2\gamma+1}}\right). \tag{2.94}$$

The above expression shows that the zeroes of the polynomials $p_n(\lambda)$ are within a distance $O(1/k)$ from the level curve (2.94). Such curve converges to $\hat{\Gamma}$ defined in (2.98) at a rate $O(\log k/k)$.

2.5.3 Zeros distribution of $p_n(\lambda)$

Let us consider the level curve $\hat{\Gamma}_r$, $0 < r \le z_0$, with $z_0 = \dfrac{t_c^2}{t^2}$

$$\hat{\Gamma}_r := \left\{\lambda \in \mathbb{C}\ s.t.\ \mathrm{Re}\,\hat{\phi}_r(\lambda) = 0,\ \ |\lambda^s - t| \le z_0 t\right\} \tag{2.95}$$

where

$$\hat{\phi}_r(\lambda) = \log(t - \lambda^s) + \frac{\lambda^s}{tz_0} - \log rt + \frac{r-1}{z_0}, \tag{2.96}$$

and r is a positive constant. We consider the usual counter-clockwise orientation for $\hat{\Gamma}_r$. These level curves consists of s close contours contained in the set D, where D has been defined in (2.3). Associated with this family of curves let us consider the measure

$$d\hat{\nu} = \frac{1}{2\pi i s}d\hat{\phi}_r(\lambda). \tag{2.97}$$

Lemma 2.3. *The a-priori complex measure $d\hat{\nu}$ in* (2.97) *is a unit positive measure on the contour $\hat{\Gamma}_r$ defined in* (2.95) *for* $0 < r \le \frac{t}{t_c}$.

Proof. Using the residue theorem, it is straightforward to check that the measure $d\hat{\nu}$ is normalised to one on any contour $\hat{\Gamma}_r$ defined in (2.95). In the post critical case such curve $\hat{\Gamma}_r$ has s connected components $\hat{\Gamma}_r^j$, $j = 0, \ldots, s-1$, each encircling the j-th root of unity of the equation $\lambda^s = t$. In the pre-critical case we denote with the same symbol $\hat{\Gamma}_r^j$ the connected components of $\hat{\Gamma}_r \setminus \{0\}$ together with $\{0\}$.

Let us consider now a parametrization $z(\tau)$ of the curve Γ_r (τ on the unit circle):

$$d\nu(z(\tau)) = \frac{d\nu(z(\tau))}{d\tau} d\tau \ .$$

Since $z(\tau)$ is a parametrisation of Γ_r, any $\hat{\Gamma}_r^j$ can be parametrised by $\lambda(\tau) := (-t(z(\tau)-1))^{\frac{1}{s}}$ with an appropriate choice of the branch of the $s-$th root. Moreover the relation $s\, d\hat{\nu}(\lambda) = d\nu(z(\lambda))$ implies that

$$d\hat{\nu}(\lambda(\tau)) = \frac{1}{s} d\nu(z(\tau)) \ ,$$

and hence

$$\frac{d\hat{\nu}(\lambda(\tau))}{d\tau} = \frac{1}{s}\frac{d\nu(z(\tau))}{d\tau} \ .$$

Since in Lemma 2.2 we proved that $d\nu(z)$ is real and positive on Γ_r, i.e.

$$\frac{d\nu(z(\tau))}{d\tau} \geq 0 \ ,$$

we have also

$$\frac{d\hat{\nu}(\lambda(\tau))}{d\tau} \geq 0 \ ,$$

i.e. $d\hat{\nu}$ is real and positive on any $\hat{\Gamma}_r^j$ and hence on the whole $\hat{\Gamma}_r$. □

Theorem 2.4. *The zeroes of the polynomials $p_n(\lambda)$ defined in (5) behaves as follows*

- *for $n = sk+s-1$ let $\omega = e^{\frac{2\pi i}{s}}$. Then $t^{\frac{1}{s}}, \omega t^{\frac{1}{s}}, \ldots, \omega^{k-1} t^{\frac{1}{s}}$ are zeroes of the polynomials p_{ks+s-1} with multipicity k and $\lambda = 0$ is a zero with multiplicity $s-1$.*
- *for $n = ks + l$, $l = 0, \ldots, s-2$ the polynomial $p_n(\lambda)$ has a zero in $\lambda = 0$ with multiplicity l and the remaining zeroes in the limit $n, N \to \infty$ such that*

  $$N = \frac{n-l}{T}$$

  accumulates on the level curves Γ_r as in (2.95) with $r = 1$ for $t < t_c$ and $r = \frac{t_c^2}{|t|^2}$ for $t > t_c$. Namely the curve $\hat{\Gamma}$ on which the zeroes accumulate is given by

  $$\hat{\Gamma}: \quad \left| (t-\lambda^s)\exp\left(\frac{\lambda^s t}{t_c^2}\right)\right| = \begin{cases} t & \text{pre-critical case } 0 < t < t_c, \\ \frac{t_c^2}{t} e^{\frac{t^2}{t_c^2}-1} & \text{post-critical case } t > t_c, \end{cases} \tag{2.98}$$

  with $|\lambda^s - t| \leq z_0 t$. The measure $\hat{\nu}$ in (2.97) is the weak $*$ limit of the normalized zero counting measure ν_n of the polynomials p_n for $n = sk + l$, $l = 0, \ldots, s-2$.

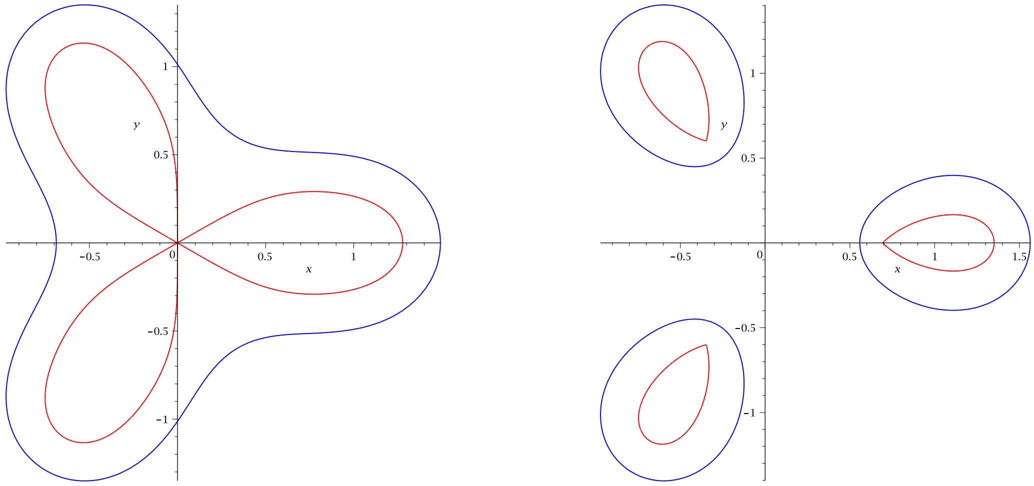


Figure 2.17: The blue contour is the boundary of the domain D defined in (2.3) and the red contour is $\hat{\Gamma}$ defined in (2.98). Here $s = 3$ and $t < t_c$ on the left figure and $t > t_c$ on the right figure. In the post-critical case, the points where the curve $\hat{\Gamma}$ is not smooth are the solutions of the equation $\lambda^s = t(1 - z_0)$, with $z_0 = t_c^2/t^2$.

Proof. From the comments after the statement of Theorem 2.2 and Theorem 2.3 it is clear from (2.92) and (2.94) that the zeroes of the polynomials $p_n(\lambda)$ accumulate in the limit $n \to \infty$ along the curve $\hat{\Gamma}$ defined in (2.98). In order to show that the measure $\hat{\nu}$ is the weak $*$ limit of the zero density ν_n we will show that

$$\lim_{n\to\infty} \frac{1}{n} \log p_n(\lambda) = \int_{\hat{\Gamma}} \log(\lambda - \xi) d\nu(\xi).$$

for λ in compact subsets of the exterior of $\hat{\Gamma}$ (namely the unbounded component of $\mathbb{C} \setminus \hat{\Gamma}$). Indeed using the relation

$$p_n(\lambda) = (-t)^k \lambda^l \pi_k(z(\lambda))$$

one has

$$\lim_{n\to\infty} \frac{1}{n} \log p_n(\lambda) = \lim_{k\to\infty} \frac{1}{ks + l} \log(\lambda^l (-t)^k \pi_k(z(\lambda))) = \frac{1}{s} \log(-t) + \frac{1}{s} \lim_{k\to\infty} \frac{1}{k} \log \pi_k(z(\lambda)) \ .$$

Using Proposition 2.3 and the relation $d\nu(z(\lambda)) = s\, d\hat{\nu}(\lambda)$ we get

$$\begin{aligned} \frac{1}{s} \log(-t) + \frac{1}{s} \lim_{k\to\infty} \frac{1}{k} \log \pi_k(z(\lambda)) &= \frac{1}{s} \log(-t) + \frac{1}{s} \int_{\Gamma} \log(z - \xi) d\nu(\xi) \\ &= \frac{1}{s} \log(-t) + \int_{\hat{\Gamma}^0} \log\left(\frac{\lambda^s - \sigma^s}{-t}\right) d\hat{\nu}(\sigma) \ , \end{aligned}$$

where $\hat{\Gamma}^j$ $(j = 0, \ldots, s-1)$ are the components of $\hat{\Gamma}$ (as defined in the proof of Lemma 2.3). Since on each $\hat{\Gamma}^j$ we have that $d\hat{\nu}$ is normalized to $\frac{1}{s}$ we have obtain

$$\begin{aligned} \frac{1}{s} \log(-t) + \int_{\hat{\Gamma}^0} \log\left|\frac{\lambda^s - \sigma^s}{-t}\right| d\hat{\nu}(\sigma) &= \int_{\hat{\Gamma}^0} \log(\lambda^s - \sigma^s) d\hat{\nu}(\sigma) = \sum_{j=0}^{s-1} \int_{\hat{\Gamma}^0} \log(\lambda - \sigma\omega^j) d\hat{\nu}(\sigma) \\ &= \sum_{j=0}^{s-1} \int_{\hat{\Gamma}^j} \log(\lambda - \sigma) d\hat{\nu}(\sigma) = \int_{\hat{\Gamma}} \log(\lambda - \sigma) d\hat{\nu}(\sigma) \ , \end{aligned}$$

with $\omega = e^{\frac{2\pi i}{s}}$ and where in the last steps we used the symmetry of $d\hat{\nu}$. Hence we have obtained the relation

$$\lim_{n\to\infty} \frac{1}{n} \log p_n(\lambda) = \int_{\hat{\Gamma}} \log(\lambda - \sigma) d\hat{\nu}(\sigma)\,,$$

for λ in compact subsets of the exterior of $\hat{\Gamma}$ (namely the unbounded component of $\mathbb{C} \setminus \hat{\Gamma}$). The proof of theorem 2.4 is then completed. $\square$

Remark. *We observe that the curve* (2.98) *in the rescaled variable* $z = 1 - \lambda^d/t$ *takes the form*

$$\Gamma : \quad \left| z e^{-\frac{1}{z_0}(z-1)} \right| = \begin{cases} 1 & \textit{pre-critical case } z_0 > 1, \\ z_0 e^{\frac{1}{z_0}-1} & \textit{post-critical case } 0 < z_0 < 1. \end{cases}$$

and it is similar to the Szegö curve [36] $\{z \in \mathbb{C}\, s.t.\ |ze^{1-z}| = 1,\ \ |z| \leq 1\}$ *that appeared in the study of the generalized Laguerre polynomials, see e.g. [28].*

From Theorem 2.4 the following identity follows immediately.

Lemma 2.4. *The measure* $\mu_{\mathcal{V}}$ *in* (2.2) *of the eigenvalue distribution of the normal matrix model and the measure* $\hat{\nu}$ *in* (2.97) *of the zero distribution of the orthogonal polynomials are related by*

$$\int_D \frac{d\mu_{\mathcal{V}}(\eta)}{\lambda - \eta} = \int_{\hat{\Gamma}} \frac{d\hat{\nu}(\eta)}{\lambda - \eta}, \quad \lambda \in \mathbb{C}\backslash D. \tag{2.99}$$

Proof. We show that the l.h.s. of (2.99) is equal to the r.h.s. Using Stokes theorem, the relation (2.4) and the residue theorem one obtains

$$\begin{aligned} \int_D \frac{d\mu_{\mathcal{V}}(\eta)}{\lambda - \eta} &= \frac{1}{2\pi i t_c^2} \int_{\partial D} \frac{\eta^{s-1}\bar{\eta}^s d\eta}{\lambda - \eta} = \frac{1}{2\pi i t_c^2} \int_{\partial D} \frac{\eta^{s-1} S(\eta) d\eta}{\lambda - \eta} = \frac{\lambda^{s-1}}{\lambda^s - t} \\ \int_{\hat{\Gamma}} \frac{d\hat{\nu}(\eta)}{\lambda - \eta} &= \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{\eta^{s-1} d\eta}{\lambda - \eta} \left(\frac{1}{\eta^s - t} + \frac{1}{z_0 t} \right) = \frac{\lambda^{s-1}}{\lambda^s - t}. \end{aligned}$$

$\square$

Remark. *The identity* (2.99) *in Lemma 2.4 seems to be a general identity in normal matrix models. It has been verified for several other potentials, see for example [1], [11],[2],[6], [25], but a proof of such result for a generic potential is still missing.*

We also observe that for the orthogonal polynomials appearing in random matrices, in some cases, the asymptotic distribution of the zeroes is supported on the so called mother body or potential theoretic skeleton of the domain D that support the eigenvalue distribution.

Chapter 3

Integrable hierarchy of conformal maps

In [42, 43] Wiegmann and Zabrodin realized that conformal maps from the exterior of the unit disc to the exterior of a simple closed analytic curve admit as functions of the exterior harmonic moments $(t_k)_{k=1}^{\infty}$ of the analytic curve the structure of an integrable hierarchy: the dispersionless Toda lattice hierarchy.

In [11] a certain class of curves is introduced and it is shown how they are connected with the dispersionless Toda lattice hierarchy. The important property of those curves is that they are uniquely characterized by their exterior harmonic moments, which play the role of the times of hierarchy. In the following we will introduce a generalization of such curves which naturally arise as boundary of supports of equilibrium measures of normal matrix models: these curves exhibits similar properties to the ones in [11] and allow a very similar analysis.

A rigorous treatment of the connection between conformal maps and Toda lattice hierarchy in full generality is given in [38, 39]. However the class of maps we are interested in allows a particularly explicit analysis, which we think is worth to mention.

3.1 Harmonic moments and Schwarz function

Consider potentials of the form

$$V(z) = |z|^{2n} + P(z) + \overline{P(z)} \ ,$$

with $\deg(P) = m < 2n$, whose equilibrium measure can be expressed as

$$d\mu(z) = \frac{1}{\pi T}\rho(z\overline{z})dA(z) = \frac{n^2}{\pi T}|z|^{2n-2}\chi_D(z)dA(z) \ .$$

As shown in Section 1.2 the conformal map characterizing the support D is given by (1.13), and hence its $n-$th power is a rational function, i.e.

$$(f(u))^n = r^n u^n \left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right) \ .$$

This property of the conformal map allows us to generalize the concept of *polynomial curve* introduced by Elbau in [11] used to describe the class of curves which bound the support of the equilibrium measure for potentials of the form

$$|z|^2 + P(z) + \overline{P(z)} \ .$$

Definition 3.1. *An* $(n,m)-$polynomial curve *is a smooth simple closed curve in the complex plane with a parametrization* $f : \{u \in \mathbb{C} : |u| = 1\} \to \mathbb{C}$ *of the form*

$$f(u) = ru\left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right)^{\frac{1}{n}}$$

with $r > 0$ *and* m *finite. The standard (counterclockwise) orientation of the unit circle in* $\mathbb{C}$ *induces an orientation on the curve. We say that a polynomial curve is* positively oriented *if this orientation is counterclockwise.*

Remark. *A* polynomial curve of degree m *as defined in [11] is a* $(1, m+1)-$*polynomial curve arrcording to Definition 3.1.*

Definition 3.2. *The* exterior harmonic moments $(t_k)_{k=1}^{\infty}$ *of a* $(n,m)-$*polynomial curve* γ

$$t_k := \frac{1}{2\pi i k}\oint_{\gamma} z^{-k}\bar{z}\rho(z\bar{z})dz \ ,$$

where

$$\rho(z\bar{z}) = n^2|z|^{2n-2} \ .$$

The interior harmonic moments $(v_k)_{k=1}^{\infty}$ *of* γ *are*

$$v_k := \frac{1}{2\pi i}\oint_{\gamma} z^{k}\bar{z}\rho(z\bar{z})dz \ .$$

It is also useful to introduce the *total mass* t_0:

$$\begin{aligned}
t_0 &= r^{2n}\frac{1}{2\pi i}\oint_{\gamma} \bar{z}^n dz^n \\
&= r^{2n}\frac{1}{2\pi i}\oint_{|u|=1} u^{-n}\left(1 + \sum_{j=1}^{m}\overline{\alpha}_j u^j\right) u^{n-1}\left(n + \sum_{j=1}^{m}\frac{(n-j)\alpha_j}{u^j}\right) du \\
&= r^{2n}\,\mathrm{res}\left[\frac{1}{v}\left(1 + \sum_{j=1}^{m}\frac{\overline{\alpha}_j}{v^j}\right)\left(n + \sum_{j=1}^{m}(n-j)\alpha_j v^j\right); u = 0\right] \\
&= nr^{2n}\left(1 + \sum_{j=1}^{m}\frac{n-j}{n}|\alpha_j|^2\right) \ .
\end{aligned} \tag{3.1}$$

We are now ready to show how $(n,m)-$polynomial curves are uniquely determined by the exterior harmonic moments $\{t_j\}_{j=1}^{m}$ and the total mass t_0

Proposition 3.1. *Let* γ *be an* $(n,m)-$*polynomial curve encircling the origin, then*

- *the exterior harmonic moments* t_k *vanish for* $k > m$
- *there exist universal polynomials* $P_{k\ell} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_{\ell-k}]$, $1 \leq k \leq \ell - 1$, *so that for* $\ell = 2, \ldots, m$

$$kt_k = nr^{2n-k}\left[\overline{\alpha}_k + \sum_{\ell=k+1}^{m} \overline{\alpha}_\ell P_{k\ell}(\alpha_1, \ldots, \alpha_{\ell-k})\right] . \tag{3.2}$$

Moreover $P_{k\ell}$ *is a weighted homogeneous polynomial of degree* $\ell - k$ *for the assignment* $\deg(\alpha_k) = k$.

Proof. Let $f : \{u \in \mathbb{C} : |u| = 1\} \to \mathbb{C}$, defined by

$$f(u) = ru\left(1 + \sum_{j=1}^{m} \frac{\alpha_j}{u^j}\right)^{\frac{1}{n}} ,$$

be a parametrization of γ, then we have

$$\begin{aligned}
kt_k :&= \frac{1}{2\pi i}\oint_\gamma z^{-k}\bar{z}\rho(z)dz = n\frac{1}{2\pi i}\oint_\gamma z^{n-1-k}\bar{z}^n dz = n\frac{1}{2\pi i}\oint_\gamma z^{-k}\bar{z}^n dz^n \\
&= r^{2n-k}\frac{1}{2\pi i}\oint_{|u|=1} \frac{u^m}{u^{k+1}}\left(1 + \sum_{j=1}^{m}\frac{\alpha_j}{u^j}\right)^{-\frac{k}{n}}\left(\sum_{j=1}^{m}\frac{\overline{\alpha}_j}{u^{m-j}} + \frac{1}{u^m}\right)\left(n + \sum_{j=1}^{m}\frac{(n-j)\alpha_j}{u^j}\right) du \\
&= nr^{2n-k}\sum_{\ell=1}^{m}\overline{\alpha}_\ell \operatorname{res}\left[\frac{v^k}{v^{\ell+1}}\left(1 + \sum_{j=1}^{m}\alpha_j v^j\right)^{-\frac{k}{n}}\left(1 + \sum_{j=1}^{m}\frac{n-j}{n}\alpha_j v^j\right); v = 0\right] .
\end{aligned}$$

This residue vanishes for $k > \ell$. The result is obtained picking the correct term in the expansion. Hence in general the exterior harmonic moments t_k vanish for $k > m$.

The existence of the polynomials $P_{k\ell}$ is clear since they are given in term of the previous expansion, and their homogeneity follows from the fact that in the expansion α_j always appears as a coefficient of v_j. ☐

Some explicit examples for the polynomials $P_{k\ell}$ are given by

$$\begin{aligned}
P_{k,k+1}(\alpha_1) &= \frac{n-k-1}{n}\alpha_1 \\
P_{k,k+2}(\alpha_1, \alpha_2) &= \frac{n-k-2}{2n^2}(2n\alpha_2 - k\alpha_1^2) \\
P_{k,k+3}(\alpha_1, \alpha_2, \alpha_3) &= \frac{n-k-3}{6n^3}(6n^2\alpha_3 - 6nk\alpha_1\alpha_2 + k(n+k)\alpha_1^3) \\
P_{k,k+4}(\alpha_1, \ldots, \alpha_4) &= \frac{n-k-4}{24n^4}(24n^3\alpha_4 - 2kn^2(\alpha_1^4 - 6\alpha_1^2\alpha_2 + 6\alpha_2^2 + 12\alpha_1\alpha_3) \\
&\quad - 3k^2n\alpha_1^2(\alpha_1^2 - 4\alpha_2) - k^3\alpha_1^4) .
\end{aligned}$$

So far we have shown how to uniquely express the exterior harmonic moments and the total mass as functions of the parameters r and $\{\alpha_j\}_{j=1}^m$: in order to prove that $\{t_k\}_{k=0}^m$ uniquely determine the curve γ we need to invert those functions.

Theorem 3.1. *Given any $(t_3, \ldots, t_m) \in \mathbb{C}^{m-2}$ and $n \in \mathbb{N}$, there exist $\delta_0, \delta_1, \delta_2 > 0$ s.t. for all $0 < t_0 < \delta_0$, $|t_1| < \delta_1$ and $|t_2| < \delta_2$, there exists a unique positively oriented $(n,m)-$polynomial curve encircling the origin with total mass t_0 and exterior harmonic moments $(t_k)_{k=1}^m$ with $t_k = 0$ for $k > m$.*

Proof. We need to invert the map $(r, \alpha_1, \ldots, \alpha_m) \to (t_0, t_1, \ldots, t_m)$ defined by (3.1) and (3.2). Defining

$$\rho := r^2 \qquad \beta_j := r^{-j+2}\alpha_j \qquad \tau_k := \frac{1}{n} r^{-2n+2} t_k$$

we get the following relations:

$$\tau_0 = \rho + \sum_{j=1}^m \frac{n-j}{n} \rho^{j-1} |\beta_j|^2$$

$$k\tau_k = \overline{\beta}_k + \sum_{\ell=k+1}^m \overline{\beta}_\ell P_{k\ell}(\beta_1, \rho\beta_2, \ldots, \rho^{\ell-k-1}\beta_{\ell-k}) \ ,$$

which define a map

$$\mathcal{F} : (\rho, \beta_1, \ldots, \beta_m) \to (\tau_0, \tau_1, \ldots, \tau_m) \ .$$

Expanding τ_k to the first order in ρ and β_1 we get

$$\tau_0 = \left(1 + \frac{n-2}{n}|\beta_2|^2\right)\rho + \ldots$$

$$k\tau_k = \overline{\beta}_k + \frac{n-k-1}{n}\overline{\beta}_{k+1}\beta_1 + \frac{n-k-2}{n}\overline{\beta}_{k+2}\beta_2\rho + \ldots \ .$$

Hence we have that $\mathcal{F}(0, 0, 2\overline{\tau}_2, \ldots, m\overline{\tau}_m) = (0, 0, \tau_2, \ldots, \tau_m)$ and the tangent map at this point $d\mathcal{F}(0, 0, 2\overline{\tau}_2, \ldots, m\overline{\tau}_m)$ sends $(\dot\rho, \dot\beta_1, \ldots, \dot\beta_m)$ to $(\dot\tau_0, \ldots, \dot\tau_m)$ with

$$\dot\tau_0 = \left(1 + 4\frac{n-2}{n}|\tau_2|^2\right)\dot\rho$$

$$k\dot\tau_k = \overline{\dot\beta}_k + \frac{n-k-1}{n}(k+1)\tau_{k+1}\dot\beta_1 + 2\frac{n-k-2}{n}(k+2)\tau_{k+2}\overline{\tau}_2\dot\rho \ .$$

For $n \geq 2$ the tangent map is always invertible and preserves the positivity of the first coordinate. For $n = 1$ we need to require $|\tau_2| < \frac{1}{2}$ in order to have these properties. Inverse function theorem guarantees that $\mathcal{F}$ has a smooth inverse in a neighborhood of $(0, 0, \tau_2, \ldots, \tau_m)$.

This means that for small enough values of $\tau_0 > 0$ and $|\tau_1|$ (and with the additional constraint $|\tau_2| < \frac{1}{2}$ if $n = 1$) there is an $(n,m)-$polynomial curve parametrized by f

$$f(u) = ru\left(1 + \sum_{j=1}^m \frac{r^{j-2}\beta_j}{u^j}\right)^{\frac{1}{n}} \qquad \text{with} \qquad \beta_j = j\overline{\tau}_j + \mathcal{O}\left(r^2\right) \ .$$

We need now to show that, for small positive values of r, f is the parametrization of a positively oriented curve which encircles the origin.

We need now to check that f is univalent, i.e. to verify the condition (see Theorem A.1 and [31])

$$\mathrm{Re}\left(u\frac{f'(u)}{f(u)}\right) > 0 \qquad \text{for } |u| \geq 1 \ ,$$

which also implies that the curve contain the origin. Indeed we have

$$\mathrm{Re}\left(u\frac{f'(u)}{f(u)}\right) = \mathrm{Re}\left(\frac{u}{n}\frac{((f(u))^n)'}{(f(u))^n}\right) = \mathrm{Re}\left(1 - \sum_{j=1}^{m} \frac{j\beta_j}{n(f(u))^n} r^{n+j-2} u^{n-j}\right) .$$

Since we can keep the absolute value of the summands as small as we need by choosing small enough r, τ_1 and τ_2 we have the desired conclusion. □

Remark. *Notice that in the case $n = 1$ the result can be made sharper (see [11]): in fact in that case we just need to require t_0 and $|t_1|$ to be small and $|t_2|$ to be bounded ($|t_2| < \frac{1}{2}$).*

3.1.1 Schwarz function

We are now going to introduce a tool which will be fundamental to show the connection between conformal maps and integrable hierarchies, namely the *Schwarz function* (for more details see [8]).

Definition 3.3. *The* Schwarz function *of a nonsingular analytic Jordan curve γ is defined as the analytic continuation (in a neighborhood of γ) of the function $S(z) = \bar{z}$ on γ.*

Let f be a parametrization of γ: using the definition of the Schwarz function S we have that in a neighborhood of the unit circle

$$S(f(u)) = \overline{f}\left(\frac{1}{u}\right) .$$

Hence it follows that in a neighborhood of γ we have

$$S(z) = \overline{f}\left(\frac{1}{f^{-1}(z)}\right) . \tag{3.3}$$

Consider now an $(n, m)-$polynomial curve γ: it is a trivial fact that the harmonic moments of γ can be written in terms of the $n-$th power of the Schwarz function. Indeed since on γ we have $\bar{z} = S(z)$ we can write

$$\begin{aligned} t_0 &= n^2 r^{2n} \frac{1}{2\pi i} \oint_\gamma z^{n-1} S^n(z) dz \\ t_k &= n^2 r^{2n} \frac{1}{2\pi i k} \oint_\gamma z^{n-1-k} S^n(z) dz \\ v_k &= n^2 r^{2n} \frac{1}{2\pi i} \oint_\gamma z^{n-1+k} S^n(z) dz . \end{aligned}$$

Redefining

$$t_k := \frac{1}{n^2} r^{-2n} t_k \qquad\qquad v_k := \frac{1}{n^2} r^{-2n} v_k$$

we can write the Laurent expansion of $S^n(z)$ for $z \to \infty$ in terms of the harmonic moments

$$S^n(z) = \sum_{k=1}^{m} k t_k z^{k-n} + \frac{t_0}{z^n} + \sum_{k=1}^{\infty} v_k z^{-k-n} = z^{-n+1} \left[\sum_{k=1}^{m} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}\right] \tag{3.4}$$

Remark. *Since we have shown that the exterior harmonic moments* $(t_k)_{k=1}^{\infty}$ *and the total mass* t_0 *characterize uniquely the* $(n,m)-$*polynomial curve, we have to consider the interior harmonic moments in the expansion* (3.4) *as functions of the* t*'s.*

3.2 The Toda lattice hierarchy

The two dimensional *Toda lattice equation* is

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} b(n) = e^{b(n+1)-b(n)} - e^{b(n)-b(n-1)} , \tag{3.5}$$

where b is a function of the continuous variables t_1 and $\tilde{t}_1$ and of the discrete variable $n \in \mathbb{N}$. In order to obtain the *dispersionless limit* of (3.5) it is useful to define

$$a(n\varepsilon, t_1, \tilde{t}_1) := b(n) - b(n-1) ,$$

and $t_0 := n\varepsilon$, so that t_0 becomes a continuous variable in the continuum limit $\varepsilon \to 0$, $n \to \infty$. By rescaling $t_1 \to t_1/\varepsilon$ and $\tilde{t}_1 \to \tilde{t}_1/\varepsilon$ equation (3.5) reduce to

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} a(n\varepsilon, t_1, \tilde{t}_1) = \frac{1}{\varepsilon^2} \left(e^{a((n+1)\varepsilon, t_1, \tilde{t}_1)} - 2e^{a(n\varepsilon, t_1, \tilde{t}_1)} + e^{a((n-1)\varepsilon, t_1, \tilde{t}_1)} \right) ,$$

which, taking the formal limit $\varepsilon \to 0$, $n \to \infty$, yields to the *dispersionless Toda equation*

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} a(t_0, t_1, \tilde{t}_1) = \frac{\partial^2}{\partial t_0^2} e^{a(t_0, t_1, \tilde{t}_1)} . \tag{3.6}$$

Equation (3.5) can be seen as an element of the so called *Toda lattice hierarchy*, introduced by Ueno and Takasaki in [41]. A review of the Toda lattice hierarchy and its dispersionless limit can be found in [37].

Definition 3.4 ([37])**.** *The* Toda lattice hierarchy *is defined by the Lax-Sato equations*

$$\varepsilon \frac{\partial L}{\partial t_k} = [M^{(k)}, L], \qquad \varepsilon \frac{\partial \tilde{L}}{\partial t_k} = [M^{(k)}, \tilde{L}], \tag{3.7}$$

$$\varepsilon \frac{\partial L}{\partial \tilde{t}_k} = [L, \tilde{M}^{(k)}], \qquad \varepsilon \frac{\partial \tilde{L}}{\partial \tilde{t}_k} = [\tilde{L}, \tilde{M}^{(k)}], \tag{3.8}$$

$k \in \mathbb{N}$, *for the difference operators*

$$L(t_0, \underline{t}, \underline{\tilde{t}}) = r(t_0, \underline{t}, \underline{\tilde{t}}) e^{\varepsilon \partial_{t_0}} + \sum_{j=0}^{\infty} a_j(t_0, \underline{t}, \underline{\tilde{t}}) e^{-j\varepsilon \partial_{t_0}} \quad \textit{and}$$

$$\tilde{L}(t_0, \underline{t}, \underline{\tilde{t}}) = \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}}) e^{-\varepsilon \partial_{t_0}} + \sum_{j=0}^{\infty} \tilde{a}_j(t_0, \underline{t}, \underline{\tilde{t}}) e^{j\varepsilon \partial_{t_0}},$$

$\underline{t} = (t_k)_{k=1}^{\infty}$, $\underline{\tilde{t}} = (\tilde{t}_k)_{k=1}^{\infty}$, *on the space of all analytic functions in* t_0, *where*

$$M^{(k)} = (L^k)_+ + \frac{1}{2}(L^k)_0 \quad \textit{and} \quad \tilde{M}^{(k)} = (\tilde{L}^k)_- + \frac{1}{2}(\tilde{L}^k)_0, \quad k \in \mathbb{N}. \tag{3.9}$$

For any operator the subscripts $+$, 0, *and* $-$ *denote its positive, constant and negative part in the* shift operator

$$e^{\varepsilon \partial_{t_0}} := \sum_{j=0}^{\infty} \frac{1}{j!} \varepsilon^j \frac{\partial^j}{\partial t_0^j}.$$

Proposition 3.2 ([37]). *Let L and $\tilde{L}$ be a solution of the Toda lattice hierarchy. Then, with $M^{(k)}$ and $\tilde{M}^{(k)}$ given by* (3.9)*, the compatibility relations*

$$\frac{\partial M^{(j)}}{\partial t_k} - \frac{\partial M^{(k)}}{\partial t_j} = \frac{1}{\varepsilon}[M^{(k)}, M^{(j)}], \tag{3.10}$$

$$\frac{\partial \tilde{M}^{(j)}}{\partial \tilde{t}_k} - \frac{\partial \tilde{M}^{(k)}}{\partial \tilde{t}_j} = \frac{1}{\varepsilon}[\tilde{M}^{(j)}, \tilde{M}^{(k)}], \tag{3.11}$$

$$\frac{\partial M^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} = \frac{1}{\varepsilon}[M^{(j)}, \tilde{M}^{(k)}], \tag{3.12}$$

$k, j \in \mathbb{N}$, for the equations (3.7) *and* (3.8) *are fulfilled.*

Proof. By defining $Q^{(k)} = L^k - M^{(k)}$ and $\tilde{Q}^{(k)} = \tilde{L}^k - \tilde{M}^{(k)}$ we get

$$\begin{aligned}\frac{\partial L_N^j}{\partial t_k} - \frac{\partial L_N^k}{\partial t_j} &= \frac{1}{\varepsilon}[M^{(k)}, M^{(j)} + Q^{(j)}] + \frac{1}{\varepsilon}[Q^{(j)}, M^{(k)} + Q^{(k)}] \\ &= \frac{1}{\varepsilon}[M^{(k)}, M^{(j)}] + \frac{1}{\varepsilon}[Q^{(j)}, Q^{(k)}].\end{aligned}$$

Adding now the positive and half of the zeroth part of this equation gives us (3.10). On the same way, by taking the negative and zeroth part of

$$\begin{aligned}\frac{\partial \tilde{L}_N^j}{\partial \tilde{t}_k} - \frac{\partial \tilde{L}_N^k}{\partial \tilde{t}_j} &= \frac{1}{\varepsilon}[\tilde{M}^{(j)} + \tilde{Q}^{(j)}, \tilde{M}^{(k)}] + \frac{1}{\varepsilon}[\tilde{M}^{(k)} + \tilde{Q}^{(k)}, \tilde{Q}^{(j)}] \\ &= \frac{1}{\varepsilon}[\tilde{M}^{(j)}, \tilde{M}^{(k)}] + \frac{1}{\varepsilon}[\tilde{Q}^{(k)}, \tilde{Q}^{(j)}].\end{aligned}$$

we obtain (3.11). Now

$$\begin{aligned}\left(\frac{\partial M^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} - \frac{1}{\varepsilon}[M^{(j)}, \tilde{M}^{(k)}]\right)_+ &= \left(-\frac{\partial Q^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} + \frac{1}{\varepsilon}[Q^{(j)}, \tilde{M}^{(k)}]\right)_+ = 0, \\ \left(\frac{\partial M^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} - \frac{1}{\varepsilon}[M^{(j)}, \tilde{M}^{(k)}]\right)_- &= \left(\frac{\partial M^{(j)}}{\partial \tilde{t}_k} - \frac{\partial \tilde{Q}^{(k)}}{\partial t_j} + \frac{1}{\varepsilon}[M^{(j)}, \tilde{Q}^{(k)}]\right)_- = 0,\end{aligned}$$

and therefore, since

$$\begin{aligned}\left(-\frac{\partial Q^{(j)}}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}^{(k)}}{\partial t_j} + \frac{1}{\varepsilon}[Q^{(j)}, \tilde{M}^{(k)}]\right)_0 &= -\frac{1}{2}\left(\frac{\partial L_N^j}{\partial \tilde{t}_k}\right)_0 + \frac{1}{2}\left(\frac{\partial \tilde{L}_N^k}{\partial t_k}\right)_0 \\ &= -\left(\frac{\partial M^{(j)}}{\partial \tilde{t}_k} - \frac{\partial \tilde{Q}^{(k)}}{\partial t_j} + \frac{1}{\varepsilon}[M^{(j)}, \tilde{Q}^{(k)}]\right)_0,\end{aligned}$$

equation (3.12) is fulfilled, too. □

Taking formally the small ε limit of the Toda lattice hierarchy, the shift operator $e^{\varepsilon\partial_{t_0}}$ is replaced by a variable u and the scaled commutator $\frac{1}{\varepsilon}[\cdot,\cdot]$ becomes a Poisson bracket with respect to the canonical variables $\log u$ and t_0. This leads us to the following definition of the *dispersionless Toda lattice hierarchy.*

Definition 3.5 ([37, 11]). *The dispersionless Toda lattice hierarchy is given by the system of equations*

$$\frac{\partial z}{\partial t_k} = \{M_k, z\}, \qquad \frac{\partial \tilde{z}}{\partial t_k} = \{M_k, \tilde{z}\},$$
$$\frac{\partial z}{\partial \tilde{t}_k} = \{z, \tilde{M}_k\}, \qquad \frac{\partial \tilde{z}}{\partial \tilde{t}_k} = \{\tilde{z}, \tilde{M}_k\},$$

$k \in \mathbb{N}$, *for functions* z *and* $\tilde{z}$ *of* u, t_0, $\underline{t} = (t_k)_{k=1}^{\infty}$, *and* $\underline{\tilde{t}} = (\tilde{t}_k)_{k=1}^{\infty}$ *of the form*

$$z(u, t_0, \underline{t}, \underline{\tilde{t}}) = r(t_0, \underline{t}, \underline{\tilde{t}})u + \sum_{j=0}^{\infty} a_j(t_0, \underline{t}, \underline{\tilde{t}})u^{-j} \quad \textit{and}$$

$$\tilde{z}(u, t_0, \underline{t}, \underline{\tilde{t}}) = \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})u^{-1} + \sum_{j=0}^{\infty} \tilde{a}_j(t_0, \underline{t}, \underline{\tilde{t}})u^{j}.$$

Denoting with the subscripts $+$, 0, *and* $-$ *the positive, constant and negative part of a function considered as power series in* u,

$$M_k = (z^k)_+ + \frac{1}{2}(z^k)_0 \quad \textit{and} \quad \tilde{M}_k = (\tilde{z}^k)_- + \frac{1}{2}(\tilde{z}^k)_0, \quad k \in \mathbb{N}, \tag{3.13}$$

and the Poisson bracket is defined as

$$\{f, g\} = u\frac{\partial f}{\partial u}\frac{\partial g}{\partial t_0} - u\frac{\partial f}{\partial t_0}\frac{\partial g}{\partial u}. \tag{3.14}$$

The compatibility relations for the dispersionless Toda lattice hierarchy are also given as the limit $\varepsilon \to 0$, $n \to \infty$ of the compatibility relations (3.10) and (3.11).

Replacing the scaled commutators $\frac{1}{\varepsilon}[\cdot, \cdot]$ with the Poisson brackets $\{\cdot, \cdot\}$ in Proposition 3.2 we can prove the following

Proposition 3.3 ([37]). *Let* z *and* $\tilde{z}$ *be a solution of the dispersionless Toda lattice hierarchy. Then* M_k *and* $\tilde{M}_k$ *given by* (3.13) *satisfy the following equations:*

$$\frac{\partial M_j}{\partial t_k} - \frac{\partial M_k}{\partial t_j} = \{M_k, M_j\} \qquad \frac{\partial \tilde{M}_j}{\partial \tilde{t}_k} - \frac{\partial \tilde{M}_k}{\partial \tilde{t}_j} = \{\tilde{M}_j, \tilde{M}_k\} \qquad \frac{\partial M_j}{\partial \tilde{t}_k} + \frac{\partial \tilde{M}_k}{\partial t_j} = \{M_j, \tilde{M}_k\},$$

with $k, j \in \mathbb{N}$.

The proof is, *mutatis mutandis*, the same as the one of Proposition 3.2.

Remark. *The two dimensional dispersionless Toda lattice equation* (3.6) *can be obtained from the compatibility equation*

$$\frac{\partial M_1}{\partial \tilde{t}_1} + \frac{\partial \tilde{M}_1}{\partial t_1} = \{M_1, \tilde{M}_1\}$$

to obtain

$$\frac{\partial^2}{\partial t_1 \partial \tilde{t}_1} \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right) = \frac{\partial^2}{\partial t_0^2}\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right),$$

hence it is easy to recover (3.6) *by setting* $a(t_0, t_1, \tilde{t}_1) = \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right)$.

The same can be done in the dispersionful case: the two dimensional Toda lattice equation can be obtained from the compatibility condition (3.12) *with* $j = k = 1$ *and by setting* $a(t_0, t_1, \tilde{t}_1) = \log\left(r(t_0, \underline{t}, \underline{\tilde{t}})\, \tilde{r}(t_0, \underline{t}, \underline{\tilde{t}})\right)$.

3.3 Schwarz function and Toda hierarchy

Consider now an $(n, m)-$polynomial curve parametrized by a conformal map of the form (1.13), uniquely characterized by the harmonic moments $(t_k)_{k=1}^{\infty}$ and the total mass t_0, and set

$$z(u, \underline{t}) = f(u) \qquad \tilde{z}(u, \underline{t}) = \overline{f}(u^{-1}) \ ,$$

where $\underline{t} := (t_k)_{k=0}^{\infty} \cup (\bar{t}_k)_{k=0}^{\infty}$. Notice that here the role of $\tilde{t}_k$'s is assumed by the complex conjugate harmonic moments $(\bar{t}_k)_{k=0}^{\infty}$. We can show that z and $\tilde{z}$ satisfy dispersionless Toda as for the case presented in [11], but with a different string equation.

Proposition 3.4. *We have the following* string equation

$$\{z^n, \tilde{z}^n\}(u, \underline{t}) = n \tag{3.15}$$

where the Poisson bracket $\{\cdot, \cdot\}$ *is the one defined in* (3.14).

Proof. Using the relation $\tilde{z}(u, \underline{t}) = S(z(u, \underline{t}), \underline{t})$, for $|u| > 1$ we get

$$\begin{aligned}
\{z^n, \tilde{z}^n\}(u, \underline{t}) = u &\left[\frac{\partial z^n}{\partial u}(u, \underline{t}) \left(\frac{\partial S^n}{\partial t_0}(z(u, \underline{t}), \underline{t}) + \frac{\partial S^n}{\partial z}(z(u, \underline{t}), \underline{t}) \frac{\partial z}{\partial t_0}(u, \underline{t}) \right) \right. \\
&\left. - \frac{\partial z^n}{\partial t_0}(u, \underline{t}) \frac{\partial S^n}{\partial z}(z(u, \underline{t}), \underline{t}) \frac{\partial z}{\partial u}(u, \underline{t}) \right] \\
&= nuz^{n-1}(u, \underline{t}) \left[\frac{\partial z}{\partial u}(u, \underline{t}) \left(\frac{\partial S^n}{\partial t_0}(z(u, \underline{t}), \underline{t}) + \frac{\partial S^n}{\partial z}(z(u, \underline{t}), \underline{t}) \frac{\partial z}{\partial t_0}(u, \underline{t}) \right) \right. \\
&\left. - \frac{\partial z}{\partial t_0}(u, \underline{t}) \frac{\partial S^n}{\partial z}(z(u, \underline{t}), \underline{t}) \frac{\partial z}{\partial u}(u, \underline{t}) \right] \\
&= nuz^{n-1}(u, \underline{t}) \frac{\partial z}{\partial u}(u, \underline{t}) \frac{\partial S^n}{\partial t_0}(z(u, \underline{t}), \underline{t}) = u \frac{\partial z^n}{\partial u}(u, \underline{t}) \frac{\partial S^n}{\partial t_0}(z(u, \underline{t}), \underline{t}) \\
&= u \frac{1}{z^n(u, \underline{t})} \frac{\partial z^n}{\partial u}(u, \underline{t}) = n + \mathcal{O}\left(\frac{1}{u}\right) \ .
\end{aligned}$$

Using now the relation $z(u, \underline{t}) = \overline{S}(\tilde{z}(u, \underline{t}), \underline{t})$, for $|u| > 1$ we get

$$\begin{aligned}
\{z^n, \tilde{z}^n\}(u, \underline{t}) = u &\left[\frac{\partial \overline{S}^n}{\partial \tilde{z}}(\tilde{z}(u, \underline{t}), \underline{t}) \frac{\partial \tilde{z}}{\partial u}(u, \underline{t}) \frac{\partial \tilde{z}^n}{\partial t_0}(u, \underline{t}) \right. \\
&\left. - \left(\frac{\partial \overline{S}^n}{\partial t_0}(\tilde{z}(u, \underline{t}), \underline{t}) + \frac{\partial \overline{S}^n}{\partial \tilde{z}}(\tilde{z}(u, \underline{t}), \underline{t}) \frac{\partial \tilde{z}}{\partial t_0}(u, \underline{t}) \right) \frac{\partial \tilde{z}^n}{\partial u}(u, \underline{t}) \right] \\
&= -u \frac{\partial \overline{S}^n}{\partial t_0}(\tilde{z}(u, \underline{t}), \underline{t}) \frac{\partial \tilde{z}^n}{\partial u}(u, \underline{t}) = -u \frac{1}{\tilde{z}^n(u, \underline{t})} \frac{\partial \tilde{z}^n}{\partial u}(u, \underline{t}) \\
&= n + \mathcal{O}(u) \ .
\end{aligned}$$

Hence we obtain $\{z^n, \tilde{z}^n\}(u, \underline{t}) = n$ by analytic continuation. $\square$

Remark. *Notice that using the properties of Poisson brackets we can write the sting equation* (3.15) *as*

$$\{z, \tilde{z}\}(u, \underline{t}) = \frac{1}{nz^{n-1}(u, \underline{t})\tilde{z}^{n-1}(u, \underline{t})} \ .$$

Proposition 3.5. *There exists a function $\Omega(z, \underline{t})$ s.t.*

$$z^{n-1}(u,\underline{t})S^n(z(u,\underline{t}),\underline{t}) = \frac{\partial \Omega}{\partial z}(z(u,\underline{t}),\underline{t}) \qquad \text{and} \qquad \log u = \frac{\partial \Omega}{\partial t_0}(z(u,\underline{t}),\underline{t}) \qquad \text{for } |u| > 1 \ ,$$

where S is the Schwarz function a curve γ. Moreover, every such function has an asymptotic expansion around $z = \infty$ of the form:

$$\Omega(z,\underline{t}) = \sum_{k=1}^{m} t_k z^k + t_0 \log z - \frac{1}{2}v_0(\underline{t}) - \sum_{k=1}^{\infty} \frac{v_k(\underline{t})}{k} z^{-k}$$

where $v_k(\underline{t})$ are the interior harmonic moments of the curve γ and

$$\frac{\partial v_0}{\partial t_0}(\underline{t}) = 2\log r(\underline{t})$$

with $z(u,\underline{t}) = r(t)u + \mathcal{O}(1)$ for $u \to \infty$.

Proof. We need to verify the compatibility condition

$$z^{n-1}(u,\underline{t})\frac{\partial S^n}{\partial t_0}(z(u,\underline{t}),\underline{t}) = \frac{\partial \log u}{\partial z} \ ,$$

which guarantees Using $\tilde{z}^n(u,\underline{t}) = S^n(z(u,\underline{t}),\underline{t})$ we easily get the result:

$$\begin{aligned}
z^{n-1}(u,\underline{t})\frac{\partial S^n}{\partial t_0}(z(u,\underline{t}),\underline{t}) &= z^{n-1}(u,\underline{t})\left(\frac{\partial \tilde{z}^n}{\partial t_0}(u,\underline{t}) - \frac{\partial S^n}{\partial z}(z(u,\underline{t}),\underline{t})\frac{\partial z}{\partial t_0}\right)\\
&= z^{n-1}(u,\underline{t})\left(\frac{\partial \tilde{z}^n}{\partial t_0}(u,\underline{t}) - \frac{\frac{\partial \tilde{z}^n}{\partial u}(u,\underline{t})}{\frac{\partial z}{\partial u}(u,\underline{t})}\frac{\partial z}{\partial t_0}(u,\underline{t})\right)\\
&= \frac{1}{\frac{\partial z}{\partial u}(u,\underline{t})}\frac{1}{nu}\left\{z^n, \tilde{z}^n\right\}(u,\underline{t}) = \frac{1}{u\frac{\partial z}{\partial u}} = \frac{\partial \log u}{\partial z} \ .
\end{aligned}$$

From the zeroth order expansion for $u \to \infty$

$$\log u = \frac{\partial \Omega}{\partial t_0}(z(u,\underline{t}),\underline{t}) = \log z - \frac{1}{2}\frac{\partial v_0}{\partial t_0}(\underline{t}) + \mathcal{O}\left(\frac{1}{u}\right) = \log u + \log r(\underline{t}) - \frac{1}{2}\frac{\partial v_0}{\partial t_0}(\underline{t}) + \mathcal{O}\left(\frac{1}{u}\right)$$

we get

$$\frac{\partial v_0}{\partial t_0}(\underline{t}) = 2\log r(\underline{t}) \ .$$

□

Proposition 3.6. *We have for $1 \le k \le m$*

$$\frac{\partial z}{\partial t_k} = \{M_k, z\} \qquad \frac{\partial \tilde{z}}{\partial t_k} = \{M_k, \tilde{z}\} \qquad \frac{\partial z}{\partial \bar{t}_k} = \{z, \tilde{M}_k\} \qquad \frac{\partial \tilde{z}}{\partial \bar{t}_k} = \{\tilde{z}, \tilde{M}_k\} \ ,$$

where

$$M_k(u,\underline{t}) := \left(z^k(u,\underline{t})\right)_+ + \frac{1}{2}\left(z^k(u,\underline{t})\right)_0 \qquad \left(\tilde{z}^k(u,\underline{t})\right)_- + \frac{1}{2}\left(\tilde{z}^k(u,\underline{t})\right)_0$$

Proof. By choosing a function Ω as in Proposition 3.5 we can define

$$M_k(u,\underline{t}) = \frac{\partial \Omega}{\partial t_k}(z(u,\underline{t}),\underline{t}) \qquad |u|>1 \ .$$

Thanks to the relation $\frac{\partial\Omega}{\partial t_0} = \log u$ we obtain

$$\begin{aligned}
\{M_k, z\}(u,\underline{t}) &= u\frac{\partial}{\partial u}\left(\frac{\partial\Omega}{\partial t_k}(z(u,\underline{t}),\underline{t})\right)\frac{\partial z}{\partial t_0}(u,\underline{t}) - u\frac{\partial}{\partial t_0}\left(\frac{\partial\Omega}{\partial t_k}(z(u,\underline{t}),\underline{t})\right)\frac{\partial z}{\partial u}(u,\underline{t})\\
&= -u\frac{\partial^2\Omega}{\partial t_k\partial t_0}(z(u,\underline{t}),\underline{t})\frac{\partial z}{\partial u}(u,\underline{t})\\
&= -u\frac{\partial z}{\partial u}(u,\underline{t})\left(\frac{\partial}{\partial t_k}\left(\frac{\partial\Omega}{\partial t_0}(z(u,\underline{t}),\underline{t})\right) - \frac{\partial^2\Omega}{\partial z\partial t_0}(z(u,\underline{t}),\underline{t})\frac{\partial z}{\partial t_k}(u,\underline{t})\right)\\
&= -u\frac{\partial z}{\partial u}\left(\frac{\partial}{\partial t_k}\log u - \frac{\partial}{\partial z}\log u\frac{\partial z}{\partial t_k}\right) = \frac{\partial z}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial t_k} = \frac{\partial z}{\partial t_k}(u,\underline{t}) \ .
\end{aligned}$$

Using $\{z,\tilde z\} = \frac{1}{nz^{n-1}\tilde z^{n-1}}$ and $z^{n-1}(u,\underline{t})\tilde z^n(u,\underline{t}) = \frac{\partial\Omega}{\partial z}(z(u,\underline{t}),\underline{t})$ we obtain

$$\begin{aligned}
\{M_k,\tilde z\}(u,\underline{t}) &= \frac{\partial^2\Omega}{\partial z\partial t_k}(z(u,\underline{t}),\underline{t})\{z,\tilde z\}(u,\underline{t}) - uz^{n-1}\frac{\partial^2\Omega}{\partial t_k\partial t_0}(z(u,\underline{t}),\underline{t})\frac{\partial\tilde z}{\partial u}(u,\underline{t})\\
&= \frac{1}{nz^{n-1}\tilde z^{n-1}}\Bigg[\frac{\partial}{\partial t_k}\left(\frac{\partial\Omega}{\partial z}(z(u,\underline{t}),\underline{t})\right) - \frac{\partial^2\Omega}{\partial z^2}(z(u,\underline{t}),\underline{t})\frac{\partial z}{\partial t_k}(u,\underline{t})\\
&\quad - nuz^{n-1}\tilde z^{n-1}\frac{\partial\tilde z}{\partial u}(u,\underline{t})\left(\frac{\partial}{\partial t_k}\left(\frac{\partial\Omega}{\partial t_0}(z(u,\underline{t}),\underline{t})\right) - \frac{\partial^2\Omega}{\partial z\partial t_0}(z(u,\underline{t}),\underline{t})\frac{\partial z}{\partial t_k}(u,\underline{t})\right)\Bigg]\\
&= \frac{1}{nz^{n-1}\tilde z^{n-1}}\Bigg[\frac{\partial}{\partial t_k}\left(z^{n-1}(u,\underline{t})\tilde z^n(u,\underline{t})\right) - \frac{\partial}{\partial z}\left(z^{n-1}(u,\underline{t})\tilde z^n(u,\underline{t})\right)\frac{\partial z}{\partial t_k}(u,\underline{t})\\
&\quad + nuz^{n-1}\tilde z^{n-1}\frac{\partial\tilde z}{\partial u}(u,\underline{t})\frac{\partial z}{\partial t_k}(u,\underline{t})\frac{\partial}{\partial z}\log u\Bigg]\\
&= \frac{1}{n\tilde z^{n-1}}\left[\frac{\partial\tilde z^n}{\partial t_k}(u,\underline{t}) - \frac{\partial\tilde z^n}{\partial u}(u,\underline{t})\frac{\partial u}{\partial z}\frac{\partial z}{\partial t_k}(u,\underline{t}) + \frac{\partial\tilde z^n}{\partial u}(u,\underline{t})\frac{\partial z}{\partial t_k}(u,\underline{t})\frac{\partial u}{\partial z}\right]\\
&= \frac{\partial\tilde z}{\partial t_k}(u,\underline{t}) \ .
\end{aligned}$$

Using this two equations we can get the following:

$$\frac{\partial M_k}{\partial u} = \frac{1}{n}\left[\frac{\partial z^n}{\partial u}\frac{\partial\tilde z^n}{\partial t_k} - \frac{\partial z^n}{\partial t_k}\frac{\partial\tilde z^n}{\partial u}\right]$$

Using $z^n(u,\underline{t}) = \overline{S}^n(\tilde z(u,\underline{t}),\underline{t})$ and the expansion of $\tilde z(u,\underline{t})$ for $u\to 0$ we get

$$\left(\frac{\partial M_k}{\partial u}(u,\underline{t})\right)_- = \frac{1}{n}\left(-\frac{\partial\tilde z^n}{\partial u}(u,\underline{t})\frac{\partial\overline{S}^n}{\partial t_k}(\tilde z(u,\underline{t}),\underline{t})\right)_- = 0$$

The asymptotic expansion for Ω gives us

$$M_k(u,\underline{t}) = \left(z^k(u,\underline{t})\right)_+ + \left(z^k(u,\underline{t})\right)_0 - \frac{1}{2}\frac{\partial v_0}{\partial t_k}(u,\underline{t})$$

From the previous equations we can get:

$$\begin{aligned}\frac{\partial M_k}{\partial t_0}(u,\underline{t}) &= \frac{1}{n}\left[\frac{\partial z^n}{\partial t_0}(u,\underline{t})\frac{\partial \tilde{z}^n}{\partial t_k}(u,\underline{t}) - \frac{\partial z^n}{\partial t_k}(u,\underline{t})\frac{\partial \tilde{z}^n}{\partial t_0}(u,\underline{t})\right] \\ &= \frac{1}{n}\left[\frac{\partial \overline{S}^n}{\partial t_0}(\tilde{z}(u,\underline{t}),\underline{t})\frac{\partial \tilde{z}^n}{\partial t_k}(u,\underline{t}) - \frac{\partial \overline{S}^n}{\partial t_k}(\tilde{z}(u,\underline{t}),\underline{t})\frac{\partial \tilde{z}^n}{\partial t_0}(u,\underline{t})\right] \\ &= \frac{1}{r(\underline{t})}\frac{\partial r}{\partial t_k}(\underline{t}) + \mathcal{O}(u) = \frac{1}{2}\frac{\partial^2 v_0}{\partial t_0 \partial t_k}(\underline{t}) + \mathcal{O}(u)\ ,\end{aligned}$$

where in the last steps we used the expansion of $\tilde{z}(u,\underline{t})$ for $u \to 0$.

Comparing with the t_0–derivative of the expression obtained before we get

$$\frac{\partial}{\partial t_0}\left(z^k(u,\underline{t})\right)_0 - \frac{1}{2}\frac{\partial^2 v_0}{\partial t_0 \partial t_k}(\underline{t}) + \mathcal{O}(u) = \frac{1}{2}\frac{\partial^2 v_0}{\partial t_0 \partial t_k}(\underline{t}) + \mathcal{O}(u)\ ,$$

and hence

$$\frac{\partial v_0}{\partial t_k}(\underline{t}) = \left(z^k(u,\underline{t})\right)_0 + \mathcal{V}(\underline{t})\ ,$$

where $\mathcal{V}$ is a function which does not depend on t_0, which can be chosen to vanish, so that we get

$$M_k(u,\underline{t}) = \left(z^k(u,\underline{t})\right)_+ + \frac{1}{2}\left(z^k(u,\underline{t})\right)_0\ .$$

In order to get the analogue results for $\tilde{M}_k$ we can set

$$\tilde{M}(u,\underline{t}) = \overline{M_k(\overline{u}^{-1},\underline{t})} \qquad \tilde{z}(u,\underline{t}) = \overline{z(\overline{u}^{-1},\underline{t})}\ .$$

□

3.4 Example: $P(z) = t\,z^n$

Let us consider now the simple case with $P(z) = z^n$: the Schwarz function can be written explicitly and hence we can verify all the statement above directly.

Using (3.3) with the conformal map obtained from (1.21), i.e.

$$f(u) = (t_c)^{\frac{1}{n}} u\left(1 + \frac{\bar{t}}{t_c}\frac{1}{u^n}\right)^{\frac{1}{n}}\ ,$$

we easily get

$$S^n(z) = \frac{tz^n - |t|^2 + t_c^2}{z^n - \bar{t}} = t + \frac{t_c^2}{z^n - \bar{t}}\ .$$

The Laurent expansion (3.4) in this case becomes simply

$$S^n(z) = t + \frac{t_c^2}{z^n} + \sum_{k=1}^{\infty}\bar{t}^k t_c^2 z^{-(k+1)n} = z^{-n+1}\left[tz^{n-1} + \frac{t_c^2}{z} + \sum_{k=1}^{\infty}\bar{t}^k t_c^2 z^{-kn-1}\right]\ .$$

Notice that the parameter t_c was obtained imposing a normalization condition: in this context we should consider it as a free parameter, more precisely we have that $t_0 = t_c^2$.

Moreover a comparison with (3.4) gives the relations $t = nt_n$ and $\bar{t} = n\bar{t}_n$. Hence, using the notation introduced before, we can write

$$S^n(z) = nt_n + \frac{t_0}{z^n - n\bar{t}_n} = z^{-n+1}\left[nt_n z^{n-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty}(n\bar{t}_n)^k t_0 z^{-kn-1}\right] ,$$

and

$$z^n(u, t_0, t_n, \bar{t}_n)) = (t_0)^{\frac{1}{2}} u^n + n\bar{t}_n \qquad \tilde{z}^n(u, t_0, t_n, \bar{t}_n) = (t_0)^{\frac{1}{2}} u^{-n} + nt_n \ .$$

With this explicit formulae we can verify directly Propositions 3.4 and 3.6. Indeed we have

$$\{z^n, \tilde{z}^n\}(u, t_0, t_n, \bar{t}_n) = u\frac{\partial z^n}{\partial u}\frac{\partial \tilde{z}^n}{\partial t_0} - u\frac{\partial z^n}{\partial t_0}\frac{\partial \tilde{z}^n}{\partial u} = nu^n (t_0)^{\frac{1}{2}}\frac{u^{-n}}{2(t_0)^{\frac{1}{2}}} + nu\frac{u^n}{2(t_0)^{\frac{1}{2}}}(t_0)^{\frac{1}{2}} u^{-n-1} = n \ .$$

In order to verify the equations (consider for simplicity the first two) of Proposition 3.6, notice that we just need to check their analogues with z^n and $\tilde{z}^n$ in place of z and $\tilde{z}$ respectively. Since in this simplified case we are considering only the times t_n and $\bar{t}_n$, this is the same as taking $M_k = 0$ for $k \neq n$, so we just need to compute

$$M_n = (t_0)^{\frac{1}{2}} u^n + \frac{1}{2} nt_n \ ,$$

in order to verify

$$\begin{aligned} \{M_n, \tilde{z}^n\} &= \left\{z^n - \frac{1}{2}n\bar{t}_n, \tilde{z}^n\right\} = \{z^n, \tilde{z}^n\} = n = \frac{\partial \tilde{z}^n}{\partial t_n} \\ \{M_n, z^n\} &= \left\{z^n - \frac{1}{2}n\bar{t}_n, z^n\right\} = \{z^n, z^n\} = 0 = \frac{\partial z^n}{\partial t_n} \ . \end{aligned}$$

Appendix A

Details about conformal maps

A.1 Univalency

Pre-critical case

Definition A.1. *A map $f : \{|u| > 1\} \to \mathbb{C}$ is* starlike *if is univalent and the compact complement of its image is star-shaped w.r.t. the origin.*

Theorem A.1 (in [31])**.** *Let $f : \{|u| > 1\} \to \mathbb{C}$, then f is starlike iff*

$$\mathrm{Re}\left(u\frac{f'(u)}{f(u)}\right) > 0 \qquad |u| > 1 \ . \tag{A.1}$$

In order to prove that our pre-critical map

$$f(u) = ru\left(1 - \frac{\alpha}{u}\right)^{\frac{d}{n}}$$

is univalent, it suffices to show that (A.1) holds, i.e.

$$\mathrm{Re}\left(1 + \frac{d}{n}\frac{\alpha}{u-\alpha}\right) > 0 \ .$$

Thus we just need to check that

$$\mathrm{Re}\left(\frac{\alpha}{u-\alpha}\right) > -\frac{1}{2} \quad \text{for } |u| > 1 \ .$$

It is easy to see that $\frac{\alpha}{u-\alpha}$ maps the exterior of the circle of radius $|\alpha|$ into the halfplane $\mathrm{Re}(z) > -\frac{1}{2}$. This proves that our conformal map is univalent in $|u| > 1$.

Post-critical case

Let us consider the map

$$f(u) = -\frac{r}{\bar{\alpha}}\frac{1-\bar{\alpha}u}{u-\alpha}u\left(1 - \frac{\alpha}{u}\right)^{\frac{d}{n}} \ .$$

We can rescale $u \to \frac{|\alpha|}{\bar{\alpha}}u$ so that

$$f(u) = -\frac{r}{\bar{\alpha}}\frac{1-|\alpha|u}{u-|\alpha|}u\left(1 - \frac{|\alpha|}{u}\right)^{\frac{d}{n}} = -\frac{r}{\bar{\alpha}}(1-|\alpha|u)\left(1 - \frac{|\alpha|}{u}\right)^{\frac{d}{n}-1} \ .$$

We need the following lemma:

Lemma A.1 (in [31])**.** *Let f be a map analytic in $\{|u| \geq 1\}$ and injective on $\{|u| = 1\}$, then f is univalent in $\{|u| \geq 1\}$.*

So we just need to prove that the image of unit circle has no self–intersections, i.e. $f(u) = f(v)$ iff $u = v$. We first look for the points $u = \mathrm{e}^{i\mu}$ and $v = \mathrm{e}^{i\nu}$ such that $|f(u)| = |f(v)|$:

$$\begin{aligned}
\left|\frac{1-|\alpha|u}{u-|\alpha|}\right| |u| \left|1-\frac{|\alpha|}{u}\right|^{\frac{d}{n}} &= \left|\frac{1-|\alpha|v}{v-|\alpha|}\right| |v| \left|1-\frac{|\alpha|}{v}\right|^{\frac{d}{n}} \\
\left|1-\frac{|\alpha|}{u}\right|^{\frac{d}{n}} &= \left|1-\frac{|\alpha|}{v}\right|^{\frac{d}{n}} \\
\left|1-|\alpha|\mathrm{e}^{-i\mu}\right|^{\frac{d}{n}} &= \left|1-|\alpha|\mathrm{e}^{-i\nu}\right|^{\frac{d}{n}} \\
\left|1-|\alpha|\mathrm{e}^{-i\mu}\right|^{2} &= \left|1-|\alpha|\mathrm{e}^{-i\nu}\right|^{2} \\
2-2|\alpha|\cos(\mu) &= 2-2|\alpha|\cos(\nu) \\
\cos(\mu) &= \cos(\nu)\ .
\end{aligned}$$

Thus $|f(\mathrm{e}^{i\mu})| = |f(\mathrm{e}^{i\nu})|$ iff $\nu = \pm\mu$.
Now we just need to show that $f(\mathrm{e}^{i\mu}) \neq f(\mathrm{e}^{-i\mu})$ for $\mu \neq k\pi$.

$$\begin{aligned}
-\frac{r}{\bar{\alpha}}(1-|\alpha|\mathrm{e}^{i\mu})\left(1-|\alpha|\mathrm{e}^{-i\mu}\right)^{\frac{d}{n}-1} &= -\frac{r}{\bar{\alpha}}(1-|\alpha|\mathrm{e}^{-i\mu})\left(1-|\alpha|\mathrm{e}^{i\mu}\right)^{\frac{d}{n}-1} \\
(1-|\alpha|\mathrm{e}^{i\mu})\left(1-|\alpha|\mathrm{e}^{-i\mu}\right)^{\frac{d}{n}-1} &= (1-|\alpha|\mathrm{e}^{-i\mu})\left(1-|\alpha|\mathrm{e}^{i\mu}\right)^{\frac{d}{n}-1}\ .
\end{aligned}$$

The l.h.s. and the r.h.s. are complex conjugates, so this equation has solution iff the image of $\mathrm{e}^{i\mu}$ through the map

$$(1-|\alpha|\mathrm{e}^{i\mu})\left(1-|\alpha|\mathrm{e}^{-i\mu}\right)^{\frac{d}{n}-1} \tag{A.2}$$

is real.
Let us change the variable $y := 1-|\alpha|\mathrm{e}^{-i\mu}$: since $|\alpha| < 1$, y is contained in the unit disk centered at 1, which is contained in the half plane $\mathrm{Re}(y) > 0$. In terms of y, (A.2) assumes the simple form

$$\frac{\bar{y}}{y} y^{\frac{d}{n}}\ . \tag{A.3}$$

The map $w = y^{\frac{d}{n}}$ sends the halfplane $\mathrm{Re}(y) > 0$ into the set

$$\left\{ w \in \mathbb{C} | -\frac{d}{2n}\pi < \arg(w) < \frac{d}{2n}\pi \right\}\ ,$$

which, since $0 < d < 2n$, is contained in $\mathbb{C} \setminus \mathbb{R}_-$.
Written in its polar form, $y = \rho\mathrm{e}^{i\lambda}$, (A.3) becomes

$$\rho^{\frac{d}{n}}\mathrm{e}^{-i\frac{2n-d}{n}\lambda}\ .$$

Thus, since $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$, the previous number is real iff $\lambda = 0$, i.e. $y \in \mathbb{R}$. This implies that $1-|\alpha|\mathrm{e}^{-i\mu} \in \mathbb{R}$, but this can happen iff $\mu = k\pi$.

Therefore Lemma A.1 gives that the post-critical map $f(u)$ is univalent in $|u| > 1$.

A.2 Analysis of the equation for the conformal radius

Let us consider equation (1.20)

$$r^{\frac{4n}{d}-2} - \frac{T}{n} r^{\frac{2n}{d}-2} + \frac{n-d}{n}\frac{d^2}{n^2} t^2 = 0 \ .$$

for the conformal radius r as a function of t. We need to show that (1.20) has a unique positive solution $r = r_0$ such that

$$|\alpha| = \frac{d}{n} t r_0^{1-\frac{2n}{d}} < 1 \ , \tag{A.4}$$

or equivalently

$$|\alpha|^2 = \frac{n}{n-d}\left(\frac{T}{n} r_0^{-\frac{2n}{d}} - 1\right) < 1 \ . \tag{A.5}$$

Solving the critical equation $|\alpha| = 1$ and using (1.20) we can obtain the critical values t_c and $r_c := r(t_c)$ given by

$$t_c = \frac{n}{d}\left(\frac{T}{2n-d}\right)^{1-\frac{d}{2n}}$$

$$r_c = \left(\frac{T}{2n-d}\right)^{\frac{d}{2n}} \ .$$

Clearly the formulae above make sense only for $d \neq n$ and $d \neq 2n$. However, these cases are trivial so we can restrict ourselves to the study of the cases $0 < d < n$ and $n < d < 2n$.

Proposition A.1. *Assume that $t < t_c$.*

- *For $0 < d < n$, Eq. (1.20) has two positive solutions $r_\pm(t)$, with*

$$0 \leq r_-(t) < r_+(t) \leq \left(\frac{T}{n}\right)^{\frac{d}{2n}} \quad \text{and} \quad r_-(0) = 0 \ , \ r_+(0) = \left(\frac{T}{n}\right)^{\frac{d}{2n}} \ .$$

  With the choice $r = r_+(t)$ the inequality (A.4) is satisfied whereas the other solution $r = r_-(t)$ is not compatible with (A.4).

- *For $n < d < 2n$ Eq. (1.20) has a unique positive solution $r_0(t)$ that is compatible with the inequality (A.4).*

Proof. Let $0 < d < n$. Consider the function

$$y(r) = r^{\frac{4n}{d}-2} - \frac{T}{n} r^{\frac{2n}{d}-2}$$

on the non-negative real axis. The only roots of $y(r) = 0$ are

$$r = 0 \quad \text{and} \quad r = \left(\frac{T}{n}\right)^{\frac{d}{2n}} \ ,$$

and $y(r)$ has a unique minimum at

$$r_{min} = \left(\frac{T}{n}\frac{n-d}{2n-d}\right)^{\frac{d}{2n}}$$

with

$$y(r_{min}) = -\frac{T}{2n-d}\left(\frac{T}{n}\frac{n-d}{2n-d}\right)^{\frac{n-d}{n}} \ .$$

Now it is easy to see that there exist precisely two solutions

$$0 < r_- < r_{min} < r_+ < \left(\frac{T}{n}\right)^{\frac{d}{2n}}$$

for $0 < t < t_{max}$ where

$$t_{max} = \frac{n^2}{d}\left(\frac{T}{n}\frac{n-d}{2n-d}\right)^{\frac{n-d}{2n}}\sqrt{\frac{T}{n(n-d)(2n-d)}} \ .$$

Since

$$\frac{t_{max}}{t_c} = \left(\frac{n}{n-d}\right)^{\frac{d}{2n}} < 1 \ ,$$

this condition is always satisfied for $0 < t < t_c$. Moreover,

$$\frac{r_c}{r_{min}} = \left(\frac{n}{n-d}\right)^{\frac{d}{2n}} > 1 \ .$$

Therefore with the choice $r = r_+$ we have

$$|\alpha|^2 = \frac{n}{n-d}\left(\frac{T}{n}r_+^{-\frac{2n}{d}} - 1\right) < \frac{n}{n-d}\left(\frac{T}{n}r_c^{-\frac{2n}{d}} - 1\right) = \frac{n}{n-d}\left(\frac{2n-d}{n} - 1\right) = 1 \ .$$

In order to prove that r_- does not satisfy (A.5) we write

$$\begin{aligned}|\alpha|^2 &= \frac{n}{n-d}\left(\frac{T}{n}r_-^{-\frac{2n}{d}} - 1\right) > \frac{n}{n-d}\left(\frac{T}{n}r_{min}^{-\frac{2n}{d}} - 1\right) \\ &= \frac{n}{n-d}\left(\frac{2n-d}{n-d} - 1\right) = \left(\frac{n}{n-d}\right)^2 > 1 \ .\end{aligned}$$

Therefore the only positive solution compatible with (A.4) is $r = r_+(t)$.

For $n < d < 2n$ the function $y(r)$ is strictly increasing with

$$y(r) \to -\infty \quad r \to 0_+ \qquad \text{and} \qquad y(r) \to \infty \quad r \to \infty \ ,$$

and therefore (1.20) has a unique solution for every value of t. Since the unique root of $y(r) = 0$ is given by

$$r = \left(\frac{T}{n}\right)^{\frac{d}{2n}} \ ,$$

for $0 < t < t_c$ we have

$$\left(\frac{T}{n}\right)^{\frac{d}{2n}} < r_0(t) < r_c$$

and hence

$$|\alpha|^2 = \frac{n}{d-n}\left(1 - \frac{T}{n}r_0^{-\frac{2n}{d}}\right) < \frac{n}{d-n}\left(1 - \frac{T}{n}r_c^{-\frac{2n}{d}}\right) = \frac{n}{d-n}\left(1 - \frac{2n-d}{n}\right) = 1 \ ,$$

as needed. □

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